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Topological Structures on Fuzzy Multisets

Jwngsar Moshahary [1]

[1] Bodoland University, Department of Mathematical Sciences,
Kokrajhar, Assam 783370, India

Abstract: *In this paper the concept of fuzzy topological space extended into fuzzy multiset topological space, closed fuzzy multiset, open fuzzy multisets, interior fuzzy multisets, closure of fuzzy multisets and their various properties discuss. Further, we extended the continuous functions to fuzzy multiset continuous function* ***and also discussed subspace of fuzzy multiset topological space.***

Keywords: Fuzzy multiset, fuzzy multiset topological space, open fuzzy multiset, closed fuzzy multiset, interior fuzzy multiset, closure fuzzy multiset, fuzzy multiset continuous function and fuzzy multiset subspace**.**

1. Introduction

The concept fuzzy set theory is introduced by L. A. Zadeh [1]. Chang [2] introduced fuzzy topological space in the year of 1968. Many author has been discuss the theory and application of fuzzy topological spaces. Blizard [3] traced multisets back to the very origin of numbers, arguing that in ancient times, the number n was often represented by a collection of n strokes, tally marks, or units. The idea of fuzzy multiset was introduced by Yager [4] as fuzzy bags. In the interest of brevity, we shall consider our attention in the basic concepts such as an open fuzzy multiset (OFMS), closed fuzzy multiset (CFMS), interior, closure and continuity of fuzzy multisets. Our notation and terminology for fuzzy multiset follows [4] and [5].

2. Preliminaries

2.1 Definition: If X is a collection of objects denoted by x then a fuzzy set A in X is a set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in X\}$$

Where $\mu_A(x)$ is called the membership function.

2.2 Definition: Let X be a non-empty set. A Fuzzy multiset (FMS) A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A : X \to Q$ where Q is the set of all crisp multisets drawn from the unit interval[0,1]. Then for any $x \in X$, the value $CM_A(x)$ is crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as decreasing ordered sequence of elements in $CM_A(x)$. It is denoted by $\left(\mu_A^1(x), \mu_A^2(x), \mu_A^3(x), \dots, \mu_A^p(x)\right)$ where $\mu_A^1(x) \geq \mu_A^2(x) \geq \mu_A^3(x) \geq, \dots, \geq \mu_A^p(x)$

2.1 Example: Let X={a,b,c,d}. A crisp multiset M is expressed as {a,b,d,a,a,b}={a,a,a,b,b,d}, i.e, $C_M(a) = 3, C_M(b) = 2, C_M(c) = 0$ and $C_M(d) = 0$

An example of fuzzy multiset A of X is

A={(a,0.1),(b,0.5),(c,0.6),(c,0.6),(a,0.2),(c,0.8)}

A membership sequence is the collection of memberships for a particular element of X which is arranged in decreasing order. In the above example, the membership sequence form of the above A is

A={(0.2,0.1)/a,(0.5)/b,(0.8,0.6,0.6)/c}

2.3 Definition: Length of an element x in an FMS A is defined as the cardinality of $CM_A(x)$ and it is denoted by

$$L(x:A)=|CM_A(x)|$$

2.4 Definition: If A and B are FMS's drawn from X then

L(x:A,B)=Max{L(x:A),L(x:B)}

we can use notation L(x) for L(x:A,B).

2.2 Example: Consider X={x,y,z,w}

A={(0.3,0.2)/x,(1,0.5,0.5)/y,(0.5,0.4,0.3,0.2)/z}

B={(0.2,0.1)/x,(1,0.3,0.2)/y,(0.2,0.1)/w} Then

L(x:A)=2,L(y:A)=3,L(z:A)=4
L(x:B)=2,L(y:B)=3,L(w:B)=2

L(x:A,B)=Max{L(x:A),L(x:B)}=Max{2,2}=2,

L(y:A,B)=3,L(z:A,B)=4,L(w:A,B)=2

3 Basic Operations On Fuzzy Multisets (FMS)

3.1 Definition: For any two FMSs A and B drawn from a decreasing X, the following operations and relations will hold. Let

$A = \{(\mu_A^1(x), \mu_A^2(x), \mu_A^3(x), \dots, \mu_A^p(x))/x: x \in X\}$

and

$B = \{(\mu_B^1(x), \mu_B^2(x), \mu_B^3(x), \dots, \mu_B^p(x))/x: x \in X\}$

1. **Inclusion**

   $A \subseteq B \Leftrightarrow \mu_A^i(x) \leq \mu_B^i(x)$ i=1,2,...,L(x),x∈X

   A=B⇔A⊆B B⊆A
2. **Complement**

   $A^c = \{(1-\mu_A^1(x), 1-\mu_A^2(x), 1-\mu_A^3(x), \dots, 1-\mu_A^p(x))/x: x \in X\}$
3. **Union A∪B**

   In A∪B the membership obtained as follows,

   $\mu_{A\cup B}^i(x) = \mu_A^i(x) \wedge \mu_B^i(x)$

   i=1,2,3,...L(x),x∈X
4. **Intersection A∩B**

   In A∩B the membership obtained as follows,

   $\mu_{A\cap B}^i(x) = \mu_A^i(x) \vee \mu_B^i(x)$

   i=1,2,3,...L(x),x∈X

Here ∨, and ∧ denotes maximum and minimum real numbers respectively.

3.1 Theorem: For any three FMS's A,B,C

1. Commutative Law

   A∪B=B∪A

   A∩B=B∩A
2. Idempotent Law

   A∪A=A

   A∩A=A
3. De Morgan's Laws

   $(A\cup B)^c = (A^c \cap B^c)$

   $(A\cap B)^c = (A^c \cup B^c)$
4. Associative Law

   A∪(B∪C)=(A∪B)∪C

   A∩(B∩C)=(A∩B)∩C
5. Distributive Law

   A∪(B∩C)=(A∪C)∩(B∪C)

   A∩(B∪C)=(A∩C)∪(B∩C)

3.2 Definition: Let X and Y be two non-empty sets and f:X→Y be a mapping.Then

1. The image of FMS A∈FM(X) under mapping f is denoted by f(A) or f[A], where

   $$CM_{f(A)}(y) = \begin{cases} v_{f(x)=y} CM_A(x); f^{-1}(y) \neq \phi \\ 0 \qquad otherwise \end{cases}$$
2. The inverse image of FMS B∈FM(Y) under the mapping f is denoted by $f^{-1}(B)$ or $f^{-1}[B]$, where $CM_{f^{-1}(B)}(x) = CM_B f(x)$. .

4 Fuzzy Multiset Topological Spaces

In this section we introduce the concept of fuzzy multiset topology (FMT). Here we extend the concept of fuzzy topological space introduced by C.L. Chang [2] to the case of fuzzy multiset topology. For this we first introduced $\tilde{0}$ and $\tilde{1}$ in a non-empty set X as follows:

4.1 Definition: Let

$\tilde{0} = \{(0,0,0 \dots,0)/x: x \in X\}$ and

$\tilde{1} = \{(1,1,1, \dots,1)/x: x \in X\}$

4.2 Definition: A fuzzy multiset topology (FMT) on X is a family Γ of fuzzy multisets (FMS) such that

1. $\tilde{0}$, $\tilde{1}$ ∈Γ
2. $F_1 \cap F_2 \in \Gamma$ for any $F_1, F_2 \in \Gamma$
3. $\cup F_i \in \Gamma$ for any arbitrary family $\{F_i : i \in I\}$ in Γ

then the pair (X,Γ) is called Fuzzy Multiset Topological Space (FMTS) and any FMS in Γ is know as and open fuzzy multiset (OFMS) in X.

4.1 Remark: The complement of an OFMS is called closed fuzzy multiset (CFMS).

4.1 Example: Let X={1,2} and defined the FMSs in X as follows,

For n∈N

$$F_n = \{(1,1/2,1/3, \dots,1/n)/1, (1/2,1/3,1/4, \dots,1/(n+1))/2\}$$

Let $\Gamma = \{\tilde{0}, \tilde{1}\} \cup F_n$

Then (X,Γ) forms an FMT.

4.3 Definition: A FMS-base for a fuzzy multiset topological space(X,Γ) is a sub collection B of Γ such that each member A of Γ can be written as $A=\vee_{j\in\wedge}A_j$, where each $A_j\in B$.

4.4 Definition: Let (X,Γ) be a FMT and A be a FMS in X. Then the closure of A is denoted by cl(A) is defined as cl(A)=∩{M: M is closed in X and A⊆M}

4.5 Definition: Let (X,Γ) be a FMT and A be a FMS in X. Then the interior of A is denoted by int(A) is defined as int(A)=∪{M: M is open in X and M⊆A}

4.6 Definition: Let (X,Γ_1) and (X,Γ_2) be two FMT's on X. Then Γ_1 is coarser(weaker) than Γ_2 if $A\in\Gamma_2$ for each $A\in\Gamma_1$.It is denoted by $\Gamma_1\subseteq\Gamma_2$.

4.1 Proposition: Let $\{\Gamma_i:i\in I\}$ be a family of FMTs on X. Then $\cap\Gamma_i$ is a FMT on X and it is coarsest FMT on X containing all Γ_i's.

Proof follows above two definitions 4.5 and 4.6.

4.2 Proposition: Let (X,Γ) be an FMT and A be an FMS in X. Then cl(A) is a CFMS.

Proof: $cl(A)=\cap\{M: M \text{ is CFMS in } X \text{ and } A\subseteq M\}$
$\Rightarrow \left(cl(A)\right)^c$ is union of all open set and hence open.
$\Rightarrow cl(A)$ is CFMS.

4.3 Proposition: Let (X,Γ) be an FMT and A be an FMS in X. Then int(A) is a OFMS.

Proof follows by definition 4.5.

4.2 Remark: From above two propositions 4.2, 4.3 and the definition 4.4, 4.5 it is clear that cl(A) is the smallest CFMS containing A and int(A) is the largest OFMS contained A.

4.4 Proposition: Let (X,Γ) be an FMT and A be FMS. Then $cl(A^c)=(int(A))^c$.

Proof: Let $A=\{(\mu_A^1(x),\mu_A^2(x),\ldots,\mu_A^p(x))/x: x\in X\}$ and
$A_i=\{(\mu_{A_i}^1(x),\mu_{A_i}^2(x),\ldots,\mu_{A_i}^p(x))/x: x\in X, i\in I\}$
Be the family of OFMSs which contained A. Then
$\mu_{A_i}^j(x)\le\mu_A^j(x); i\in I; j=1,2,3,\ldots,p$
And $int(A)=\{(\vee\mu_{A_i}^1(x),\vee\mu_{A_i}^2(x),\ldots,\vee\mu_{A_i}^p)/x: x\in X, i\in I\}\Rightarrow\left(int(A)\right)^c=\{(1-\vee\mu_{A_i}^1(x),1-\vee\mu_{A_i}^2(x),\ldots,1-\vee\mu_{A_i}^p)/x: x\in X, i\in I\}$ (1)

Now $A^c=\{(1-\mu_A^1(x),1-\mu_A^2(x),\ldots,1-\mu_A^p(x))/x: x\in X\}$ From (1) it is clear that $\{A_i^c: i\in I\}$ is the family of CFMSs containing A^c

$$\Rightarrow cl(A^c)=\{\left(\wedge\left(1-\mu_A^1(x)\right),\wedge\left(1-\mu_A^2(x)\right),\ldots,\wedge\left(1-\mu_A^p(x)\right)\right)/x: x\in X\}$$

$$\Rightarrow cl(A^c)=\{\left(\left(1-\vee\mu_{A_i}^1(x)\right),\left(1-\vee\mu_{A_i}^2(x)\right),\ldots,\left(1-\vee\mu_{A_i}^p(x)\right)\right)/x: x\in X, i\in I\} \quad (2)$$

From (1) and (2)

$$cl(A^c)=\left(int(A)\right)^c.$$

4.5 Proposition: Let (X,Γ) be FMT and A be an FMS. Then $int(A^c)=(cl(A))^c$

Proof is similar to Proposition 4.4.

4.6 Proposition: Let (X,Γ) be FMT and A and B are OFMSs.Then the following properties hold.

1. int(A)⊆A
2. A⊆cl(A)
3. A⊆B⇒int(A)⊆int(B)
4. A⊂B⇒cl(A)⊆cl(B)
5. $\text{int}(\tilde{1})=\tilde{1}$
6. $\text{int}(\tilde{0})=\tilde{0}$

Proof follows from the definition.

4.7 Proposition: Let (X,Γ) be FMT and A be an FMS. Then A is CFMS if and only if A=cl(A)

Proof. Let A is CFMS. Then by proposition 4.6(2) we have
$A\subseteq cl(A)$. (3)
Since A is CFMS,
$cl(A)\subseteq A$. (4)
From (3) and (4) $cl(A)=A$
Converse follows from the definition.

4.8 Proposition: Let (X,Γ) be FMT and A be an FMS. Then A is OFMS if and only if A=int(A)

Follows from definition.

5 Continuous functions

In this section, we define fuzzy multiset continuous function and equivalence relation of fuzzy multiset continuous function.

5.1 Definition: Let (X,Γ) and (X,Φ) be two FMTs. A function f:X→Y is said to be continuous if and only if inverse image of each OFMS in Φ is an OFMS in Γ.

5.1 Theorem : Let (X,Γ) and (X,Φ) be two FMTs. A function f:X→Y is said to be continuous if and only if inverse image of each CFMS in Φ is a CFMS in Γ

Proof: Let f be continuous function and C be CFMS in Φ.
To prove $f^{-1}(C)$ is closed, it is enough to show that $\left(f^{-1}(C)\right)^c$ is open.
Now the inverse image of FMS C in Y under the mapping f is denoted by $f^{-1}(C)$ where $CM_{f^{-1}(c)}(x) = CM_C f(x)$.
$\Rightarrow f^{-1}(C) = \{(CM_c f(x))/x : x \in X\}$.
$\Rightarrow \left(f^{-1}(C)\right)^c = \{1 - CM_C f(x))/x : x \in X\}$. (5)
Now $f^{-1}(C^c) = \{(CM_{C^c} f(x))/x : x \in X\}$
$\Rightarrow f^{-1}(C^c) = \{(1 - CM_c f(x))/ x : x \in X\}$. (6)
Where $y = f(x)$
From (5) and (6)
$f^{-1}(C^c) = \left(f^{-1}(C)\right)^c$. (7)
Since f is continuous and C^c is open by definition of continuous function $f^{-1}(C^c)$ is open and hence $\left(f^{-1}(C)\right)^c$ is open.
Now assume that the inverse image of each CFMS in Φ is a CFMS in Γ
To prove that f is continuous, it is enough to prove $f^{-1}(O)$ is open for every OFMS in Φ.
$\Rightarrow O$ is an OFMS $\Rightarrow O^c$ CFMS
$\Rightarrow f^{-1}(O^c)$ is CFMS by assumption. Since (7) is true for any CFMS, $\left(f^{-1}(O)\right)^c$ is CFMS and hence $f^{-1}(O)$ is OFMS . Hence f is continuous

5.2 Theorem: Let (X,Γ) and (X,Φ) be two FMTs. A function f:X→Y is said to be continuous if and only if each FMS A in X, f(cl(A))⊆cl(f(A)).

Proof: Let us assume that f is continuous.
For any FMS A in X, $f(A) \subseteq cl(f(A))$ by proposition 4.8.2

$$\Rightarrow A = f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A))$$

Since f is continuous and $cl(A)$ is closed , $f^{-1}(cl(f(A))$ is closed.
$\Rightarrow cl(A) \subseteq f^{-1}\left(cl(f(A))\right)$, since $cl(A)$ is the smallest CFMS contains A.
Conversely assume that the given condition is true.
To prove f is continuous, Let C be a CFMS in Φ
Then by assumption, $f\left(cl(f^{-1}(C))\right) \subseteq cl\left(f(f^{-1}(C))\right) = cl(C) = C$
Thus $cl(f^{-1}(C)) \subseteq f^{-1}(C)$
By the proposition 22(2), $f^{-1}(C) \subseteq cl(f^{-1}(C))$
$\Rightarrow f^{-1}(C) = cl(f^{-1}(C))$.
$\Rightarrow f^{-1}(C)$ is CFMS
$\Rightarrow f$ is continuous.

5.3 Theorem: Let (X,Γ) and (X,Φ) be two FMTs. A function f:X→Y is said to be continuous if and only if each FMS B in Y, $cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Proof follows from theorem 5.2.

5.4 Theorem: Let (X,Γ) and (X,Φ) be two FMTs. A function f:X→Y is said to be continuous if and only if each FMS A in X, $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$.

Proof: Let us assume that f is continuous.
For any FMS A in X, $int(f(A)) \subseteq f(int(A))$ by proposition 4.8.3
$\Rightarrow f^{-1}\left(int(f(A))\right) \subseteq int(A)$.
Since f is continuous and $int(f(A))$ is OFMS, $f^{-1}(int(f(A))$ is OFMS.
But $int(A)$ is the largest OFMS containing A.
$\Rightarrow f^{-1}(int(f(A)) \subseteq int(A)$.
$\Rightarrow int(f(A)) \subseteq f(int(A))$.
Conversely assume that the given condition is true.
To prove f is continuous, let O be an OFMS in Φ
Then by assumption, $int(f(f^{-1}(O)) \subseteq f\left(int(f^{-1}(O))\right)$
$\Rightarrow int\ (O) = f(int(f^{-1}(O))$.
$\Rightarrow O \subseteq f(int(f^{-1}(O))$ By proposition 4.8
$\Rightarrow f^{-1}(O) \subseteq int(f^{-1}(O)$.
Again by proposition 4.8.1 $int(f^{-1}(O)) \subseteq f^{-1}(O)$
$\Rightarrow int(f^{-1}(O)) = f^{-1}(O)$.
$\Rightarrow f^{-1}(O)$ is OFMS
$\Rightarrow f$ is continuous.

5.5 Theorem: Let (X,Γ) and (X,Φ) be two FMTs. A function f:X→Y is said to be continuous if and only if each FMS B in Y, $f^{-1}(int(B)) \subseteq int(f^{-1}(B))$.

Proof follows from theorem 5.4.

6 Fuzzy Mulitset Subspace Topology

In this section, we give the definition of subspace of fuzzy multiset topological space and example.

6.1 Definition: Let (X,Γ) and (Y,Φ) be two FMTs. The topological space Y is called subspace of the topological space X if Y⊆Y and OFMS precisely the subsets O' of the form

O'=O∩Y

for some OFMS O of X. Here we may say each OFMS O' of Y is the restriction to Y of an OFMS O of X. O' is also called relative OFMS in Y.

6.1 Example: Let X={1,2} and defined the FMSs in X as follows,

For n∈N

$F_n = \{(1,1/2,1/3,\dots,1/(n-1))/1, (1/2,1/3,1/4,\dots,1/n)/2\}$Let $\Gamma_1 = \{\tilde{0}\ , \tilde{1}\} \cup F_n$.

Then (X,Γ_1) forms a subpace FMT of example 4.1.

6.1 Theorem: Let (X,Γ) be FMT and let Y be a subset of Y. Define the collection Φ of subsets of Y is the collection of subsets O' of Y of the form

O'=O∩Y

where O is an OFMS in (X,Γ), Then (Y$\tilde{\cup}$1,Φ) is an FMT and a subspace of (X,Γ).

Proof: We have $\tilde{0} = \tilde{0} \cap Y$
Let $O_1', O_2', O_3', \dots, O_n' \in \Phi$
Where $O_i' = O_i \cap Y$ for some $O' \in \Gamma$
Then $O_1' \cap O_2' \cap \dots \cap O_n' = (O_1 \cap O_2 \cap \dots \cap O_n) \cap Y$
Finally, Suppose that for each $\alpha \in I, O_\alpha' \in \Phi$
Thus for each $\alpha \in I, O_\alpha' = O_\alpha \cap Y$ for some $O_\alpha \in \Gamma$
Then $\cup\ O_\alpha' = \cup\ (O_\alpha \cap Y) = \cup\ O_\alpha \cap Y \in \Phi$ Since $\cup\ O_\alpha \in \Gamma$
Hence $(Y \cup \tilde{1}, \Phi)$ is FMT and therefore a subspace of (X, Γ).

7 Conclusion

In this work we studied the topological structure of Fuzzy Multisets. We introduced the concept of fuzzy multiset topological space, open fuzzy multisets, closed fuzzy multisets, interior of fuzzy multiset, closure of fuzzy multisets and discussed various properties. We discussed continuous functions and various properties in term of Fuzzy multisets.

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