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Transmission Dynamics of Measles: A Mathematical Model

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Abstract:- In this paper, a mathematical model has been studied the stability of the disease free and endemicequilibrium point and their linear stability analysis have been conducted. The model it has been shown that the disease free and also the endemic equilibrium point are linearly asymptotically stable. From the stability of disease free equilibrium point it can be shown, disease will not spread in the population and the endemic quilibrium point it can be conclude that disease will remain in the population.

Introduction :- A acute highly infectious disease of childhood caused by specific virus of the group myxovirus.it is clinically characterized by fever and catarrhal, system of the upper respiratory react (Corza, Cough), followed by a typical rash. There are many host factors. Affect virtually everyone infancy or childhood between 6 month and 3 year of age. No age is immune of there was no previous immunity. Second attack are rare. Infants are protected by maternal antibodies beyond 9 month, Measles tends to very serve in the malnourished child mortality up to 400 times higher than in well nourished children having measles. Transmission occurs directly from person to person mainly by droplet infection and droplet nuclei. Measles is best prevented by active immunization. Only live attended vaccines are recommended for use, they are both safe and effective. The need for immunoglobulin is now much reduced because of availability of an effective live attenuated vaccine. The main purpose of this paper is to construct a mathematical models to study the transmission dynamics of measles.

BASIC ASSUMPTION AND MATHEMATICAL MODEL: - In the formulation of the proposed model the underlying population has been divided into four compartments susceptible (S), exposed (E), infected (I) and removed (R). It is assumed that vaccine efficacy is related death with this assumption the mathematical model has been constructed which being given by the following system of none. Linear ordinary differential equations Λ denots the recruitment rate β, transmission rate ξ is vaccination rate, μ is nature death rate, γ is rate at which exposed population become infevtive, η denotes recovery rate.

Model:

$$\frac{dS}{dt} = A - \beta SI - \xi S - \mu S \qquad \dots\dots(1)$$

$$\frac{dE}{dt} = \beta SI - \gamma E - \mu E \qquad \ldots\ldots(2)$$

$$\frac{dI}{dt} = \gamma E - \mu I - (\eta + \xi)I \qquad \ldots\ldots(3)$$

$$\frac{dR}{dt} = (\eta + \xi)I - \mu R \qquad \ldots\ldots(4)$$

Disease free equilibrium point :- E_1 $(\overline{S}, \overline{E}, \overline{I}, \overline{R})$ on solving the above equation we get,

Where, $\overline{S} = \frac{\Lambda}{\xi+\mu}, \overline{E} = 0, \overline{I} = 0, \overline{R} = 0$

Endemic equilibrium point:-

On solving the above equation we get the following endemic equilibrium point

$E_2(S^*, E^*, I^*, R^*)$

Where

$$E^* = \frac{\Lambda}{\gamma + \mu} - \frac{(\mu + \eta + \xi)(\xi + \mu)}{\gamma\beta}$$

$$I^* = \frac{\gamma\Lambda}{(\mu + \eta + \xi)(\gamma + \mu)} - \frac{(\xi + \mu)}{\beta}$$

$$R^* = \frac{\gamma(\eta + \xi)\Lambda}{\mu(\mu + \eta + \xi)(\gamma + \mu)} - \frac{(\xi + \mu)(\eta + \xi)}{\mu\beta}$$

Linearly stability analysis of disease free equilibrium point:-

Community matrix for the system (1) to (4) is given by

$$\begin{bmatrix} -\beta\bar{I} - \xi - \mu & 0 & -\beta\bar{S} & 0 \\ \beta\bar{I} & -\gamma - \mu & \beta\bar{S} & 0 \\ 0 & \gamma & -\mu - \eta - \xi & 0 \\ 0 & 0 & \eta + \xi & -\mu \end{bmatrix}$$

Community matrix disease free equilibrium point is

$$B = \begin{bmatrix} -(\xi + \mu) & 0 & -\beta\bar{S} & 0 \\ 0 & -(\gamma + \mu) & \beta\bar{S} & 0 \\ 0 & \gamma & -(\mu + \eta + \xi) & 0 \\ 0 & 0 & \eta + \xi & -\mu \end{bmatrix} =$$

Characteristic equation of above matrix is

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$$\begin{vmatrix} -(\xi+\mu+\lambda) & 0 & -\beta\bar{S} & 0 \\ 0 & -(\gamma+\mu+\lambda) & \beta\bar{S} & 0 \\ 0 & \gamma & -(\mu+\eta+\xi+\lambda) & 0 \\ 0 & 0 & \eta+\xi & -(\mu+\lambda) \end{vmatrix} = 0$$

After solving we get

$$(\xi+\mu+\lambda)(\mu+\lambda)\left[\lambda^2+(2\mu+\eta+\gamma+\xi)\lambda+(\mu+\eta+\xi)(\gamma+\mu)-\beta\overline{S}\gamma\right]=0$$

therefore

$\lambda = -(\xi+\mu), \mu$ and other two roots are given by the following quadratic equation

$$\lambda^2+(2\mu+\eta+\gamma+\xi)\lambda+[(\mu+\eta+\xi)(\gamma+\mu)-\beta\overline{S}\gamma=0$$

Substituting the value of $\overline{S}$, we have

$$\lambda^2+(2\mu+\eta+\gamma+\xi)\lambda+(\mu+\eta+\xi)(\gamma+\mu)-\frac{\beta\gamma\Lambda}{\xi+\mu}]=0$$

$$\lambda^2+a_1\lambda+a_2=0 \ldots\ldots\ldots..(5)$$

Here

$$a_1 = 2\mu+\eta+\gamma+\xi$$

$$a_2 = (\mu+\eta+\xi)(\gamma+\mu)-\frac{\beta\gamma\Lambda}{\xi+\mu}$$

Since $a_i > 0$,

Hence by Hurwitz's condition education (5) has negative real roots or have at least negative real part if $a_2 > 0$, we get $\frac{\beta\gamma\Lambda}{(\mu+\eta+\xi)(\gamma+\mu)(\xi+\mu)} < 1$

Hence system (1)to (4) is linearly asymptotically stable if $\frac{\beta\gamma\Lambda}{(\mu+\eta+\xi)(\gamma+\mu)(\xi+\mu)} < 1$

and unstable if $\frac{\beta\gamma\Lambda}{(\mu+\eta+\xi)(\gamma+\mu)(\xi+\mu)} > 1$

Linear stability analysis for endemic equilibrium point :-

For linear sing system we take following transformations

$$S = S^* + n_1$$

$$E = E^* + n_2$$

$$I = I^* + n_3$$

$$R = R^* + n_4$$

Substituting these value in (1)-(4) we get

$$\frac{dn_1}{dt} = (\beta I^* + \xi + \mu)n_1 - \beta S^* n_3 \dots \dots \dots . (6)$$

$$\frac{dn_2}{dt} = \beta S^* n_3 + \beta I^* n_1 - (\gamma + \mu)n_2 \dots \dots \dots . (7)$$

$$\frac{dn_3}{dt} = \gamma n_2 - (\mu + \eta + \xi)n_3 \dots \dots \dots . (8)$$

$$\frac{dn_4}{dt} = (\eta + \xi)n_3 - \mu n_4 \dots \dots \dots . (9)$$

Now we take a positive definite function V such that $V = \frac{n_1^2}{2} + A_1 \frac{n_2^2}{2} + A_2 \frac{n_3^2}{2} + A_2 \frac{n_4^2}{2} \dots \dots \dots \dots (10)$

Where A*i, i=1,2,3* are arbitrary positive constants, differentiability equation (10) with respect to $\frac{dv}{dt}$ is obtain by using system of linear equation as

$\frac{dv}{dt} = [-(\beta I^* + \xi + \mu)n_1^2 + \beta S^* n_1 n_2 - A_1 \beta I^* n_1 n_2 + A_1(\gamma + \mu)n_2^2 -$

$(A_1 \beta S^* + A_2 \gamma)n_2 n_3 + A_2(\mu + \eta + \xi)n_3^2 - A_3(\eta + \xi)n_3 n_4 + A_3 \mu n_4^2]$

$$\frac{dv}{dt} = -[C_{11}n_1^2 + C_{13}n_1 n_2 + C_{23}n_2 n_3 + C_{12}n_1 n_2 + C_{22}n_2^2 + C_{33}n_3^2 + C_{34}n_3 n_4 + C_{44}n_4^2$$

Where,

$C_{11} = \beta I^* + \xi + \mu$

$C_{22} = A_1(\gamma + \mu)$

$C_{13} = \beta S^*$

$C_{23} = A_1 \beta S^* + A_2 \gamma$

$C_{33} = \mu + \eta + \xi$

$C_{12} = A_1 \beta I^*$

$C_{34} = (\eta + \xi)A_3$

$C_{44} = A_3 \mu$

$$\frac{dv}{dt} = -[\left\{\frac{C_{11}n_1^2}{2} - C_{12}n_1 n_2 + C_{22}n_2^2\right\} + \left\{\frac{C_{11}n_1^2}{2} - C_{13}n_1 n_3 + \frac{C_{33}n_3^2}{2}\right\} + \{C_{33}n_3^2 - C_{34}n_3 n_4$$

$+C_{44}n_4^2 \dots \dots . . (11)$

Using Sylvester criterion in equation (11)t can be shown that $\frac{dv}{dt}$ is negative definite following condition satisfied.

$$\frac{1}{2}C_{11}C_{22} - \frac{1}{4}C_{12}^2 > 0$$

$$\frac{1}{4}C_{11}C_{33} - \frac{1}{4}C_{13}^2 > 0$$

$$\frac{1}{2}C_{22}C_{33} - \frac{1}{4}C_{23}^2 > 0$$

$$C_{33}C_{44} - \frac{1}{4}C_{34} > 0$$

Conclusion: - The disease free and endemic equilibrium point of the model have been obtained and their linearly analysis have been conducted. For the model it has been shown that the disease free and also endemic equilibrium point are linearly asymptotically stable from the stability of disease free equilibrium points it can be shown that the disease will not spread in the population if the condition for the stability of disease free equilibrium point satisfied. From the stability of endemic equilibrium point it can be conclude that disease will remain in the population if the conditions for stability are satisfied. From the values of equilibrium point it can be observed that if the rate of vaccination tends to infinity then the population of infective becomes zero. Thus with very high rate of vaccination disease will die out in the population.

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