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Principles of Optimization Techniques to Combinatorial Optimization Problems and Decomposition

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Abstract

This paper focused on the fundamental concept and solution approaches to the combinatorial optimisation techniques. The construct of the effective methods of this paper are based on the integrations of Constraints Programming (CP), Integer Programming (IP) and local search (LS) to tackle combinatorial optimization problem from different application areas like the nurse scheduling and the portfolio selection problems. These techniques demonstrate the effectiveness of the method as well as knowledge of the quality of the solution.

1. Introduction

The combinatorial optimization has been the subject of an enormous amount of research, fundamental concepts and solution approaches to the combinatorial optimisation problems. Initially we give the definitions of the general optimisation problem and the general combinatorial optimisation problem. Then we define the scope of the research in this thesis by listing several important classes of combinatorial optimisation problems. These particular classes of combinatorial optimisation problems capture the basic structures of the two application problems that will be extensively investigated in this thesis. Hereafter, the term “combinatorial optimisation problems” refer to this list of particular problems.

After that, we summarize the solution approaches to the combinatorial optimisation problems by generally categorising the techniques into two groups: exact solution approaches and heuristic solution approaches. Next, we examine the Constraint Programming techniques to solve the combinatorial optimisation problems. The important concepts and techniques in Constraint

Programming are introduced. Continuing that, present another exact solution approach Integer Programming and the related techniques from Operational Research. Then we present decomposition methods and the corresponding solution approaches. These methods include domain independent, general decomposition methods as well as ones related with the application problems we will tackle in this these. We review the current mainstream integration approaches based on Constraint Programming, Integer Programming and local search. We do not intend to give an exclusive review of the integration methods. We focus on the integration methods related with the two application problems to tackle the model.

2. Combinatorial optimisation problems

2.1 The optimisation problem

In mathematics and computer sciences, an **optimisation problem**, or a **mathematic program**, is the problem of finding the best solution from all the feasible solutions [1]. More formally, an optimisation (minimization) problem can be stated as:

$$\min f(X)/X \in F \subset R^n \quad \text{-----------------------} \quad (1)$$

where $\mathbf{x} \in \mathbf{R^n}$ is the vector of problem variables, **R** denotes the real number, $\mathbf{R^n}$ denotes an *n*-dimensional vector space over **R** , *F* is the feasible region (the set of all feasible solutions), and *f*: *F* → **R** is the objective function. Every $\mathbf{x} \in F$ is called a feasible solution to (1). If there is a $\mathbf{x}^* \in F$ satisfying:

$$f(\mathbf{x}^*) \leq f(\mathbf{x}), \forall \mathbf{x} \in F$$

then **x*** is called the (global) optimal solution and *f* (**x***) is called the (global) minimum with regard to (1). Equivalently, an optimisation problem can be stated as follows where $\mathbf{x} \in F$ is explicitly expressed by constraint (2) and (3)

$$\min f(\mathbf{x})$$

$$S.T.C \quad g_i(X) \geq 0\,; i = 1 \ldots \ldots n \quad \text{------------------------} \ (2)$$

$$h_j(X) = 0\,; j = 1 \ldots \ldots m \quad \text{------------------------} \ (3)$$

Where *i g* and *j h* are the functions → **R** , and (2), (3) represent the *constraints* of the optimisation problem.

2.2 The combinatorial optimisation problem

When an optimisation problem has a ***finite*** number of feasible solutions, the problem is called **combinatorial optimisation problem [8]**. Several important classes of combinatorial optimisation problems will be extensively investigated in this thesis. These problems capture the basic structures of the two application problems we will tackle in this thesis. They are listed as follows:

- Linear Program: a combinatorial optimisation problem is a Linear Program if the objective function *f* in (2-1) and constraints *i g* , *j h* in (2) and (3) are the linear functions.
- Finite domain optimisation problem: a combinatorial optimisation problem is a finite domain optimisation problem if the domain of variable **x** is a finite set: xi∈ [*ai*

$,bi$],i = 1...n . In this thesis, the Constraint Satisfaction Problem and Constraint Optimisation Problem in Constraint Programming paradigm are finite domain optimisation problems.

- Integer Program: If the unknown variables are all required to be integers, then the problem is called an Integer Program or Integer Linear Program.
- Quadratic Program: If the objective function f is a quadratic function and constraints g_i, h_j are linear functions, the problem is a quadratic program. When some or all of the variables are required to be integers, the problem is called Mixed Integer Quadratic Program or Integer Quadratic Program. In this thesis, we only focus on the convex quadratic objective function.

3. Constraint Programming

In this paper, the term of **Constraint Programming** (CP) refers to the techniques that are used to represent and solve the Constraint Satisfaction Problem and Constraint Optimisation Problem arising from Artificial Intelligence. This section gives a brief introduction and basic notation of CP. A large part of this section is written based on the books [1] and [2].

Definition 1 (Variable and domain): Let x be a variable. The **domain** of x is a set of values that can be assigned to x. A single value is assigned to a variable. In this thesis we only consider the variables with finite domains.

Definition 2 (Constraint): Consider a finite sequence of variables $\boldsymbol{X} = x_1, x_2 \ldots x_n$ where $n > 0$, with respective domains $\boldsymbol{D} = D_1, D_2 \ldots D_n$ such that $x_i \in D_i$ for all i. A **constraint** C on $\boldsymbol{X}$ is defined as a subset of the Cartesian product of the domains of the variables in $\boldsymbol{X}$,

i.e. $C \subseteq D_1 \times D_2 \times \ldots \times D_n$. A constraint C is called a *unary constraint* if it is defined on one variable. A constraint C is called a *binary constraint* if it is defined on two variables. If C is defined on more than two variables, we call it a *global constraint*.

Definition 3 (Constraint Optimisation Problem): Often we want to find a solution to a CSP that is optimal with respect to a certain criteria. A **Constraint Optimisation Problem** (COP) is a CSP$(\boldsymbol{X}, \boldsymbol{D}, \boldsymbol{C})$ where $\boldsymbol{D} = D_1, D_2 \ldots D_n$, together with an objective function $f: D_1 \times D_2 \times \ldots D_n \rightarrow \mathbf{R}$ to be optimised. An optimal solution to a constraint optimisation problem is a solution to P that is optimal with respect to f. The objective function value is often represented by a variable z, together with maximizing z or minimizing z for maximization or a minimization problem, respectively. In CP, the goal is to find a solution (or all solutions) to a given CSP, or an optimal solution (or all optimal solutions) to a given COP. The solution process interleaves *constraint propagation* or *propagation* in short, and *search*

4. Operational Research techniques

Instead of following a definition, we will use the term Operational Research to specify a particular set of *methods* and *solution techniques* for the combinatorial optimisation problems. This set includes for example the techniques from Linear Programming, Integer Programming and Convex Quadratic Programming.

4.1 Linear Programming

There are many textbooks on Linear Programming and Integer Linear Programming .A very good introduction to Linear Programming and Integer Programming are given by Wolsey and Nemhauser [8].

Linear Program: A Linear Program (LP) problem is characterized by a linear objective function in decision variables and by constraints described by linear inequalities or equations:

$$\min \ c_1x_1 + \text{-----------} c_nx_n$$

$$S.T.C \ a_{11}x_1 + \text{---------} a_{1n}x_n = b_1$$

$$a_{21}x_1 + \text{---------} a_{2n}x_n = b_2$$

$$a_{m1}x_1 + \text{---------} a_{mn}x_n = b_n$$

where **x** $\in \boldsymbol{R^n}$ is the vector of decision variables, **c**$\in \boldsymbol{R^m}$ is the cost coefficient vector, **b**$\in \boldsymbol{R^m}$ is the constraint vector and *A* is the constraint coefficient matrix with elementsa_{ij} . For simplicity, we assume $n \geq m$, and the columns of *A* are indexed by the set *I*= {1,…*n*}.

Let A_B be a basis of *A*, i.e. a non-singular square sub matrix of *A*, where the set *B* indexes over the columns. Let A_N be the sub matrix of *A* indexed by the columns in *N*=*I/B*. Then the set of constraints *Ax* = *b* can be written as:

$$A_Bx_B + A_Nx_N = b$$

A solution to this equation is given by $x_B = A_B{}^{-1}b$ and $x_B = 0$. This solution is called a *basic* solution, and it is feasible if $A_B{}^{-1}b \geq 0$. The vector x_B contains the basic variables and the vector x_N constrains the non-basic variables. The **reduced cost** vector c^{-T} is defined as:

$$c^{-T} = c^T - c^T{}_{\boldsymbol{B}}A_B{}^{-1}A$$

The importance of the reduced cost vector is described by the following fundamental theorem: $x = (x_B, x_N)$ is an optimal solution if and only if $\tilde{c} \geq 0$.

4.2 Integer Programming Problem

A Linear Programming Problem in which over all or some of the decision variables are constrained to assume non-negative values is called an Integer Programming Problem .This type of problem is of particular importance in business and industry, where quite often, the fractional solutions and unrealistic because the unit are not divisible .The integer solution to a problem can, however, be obtained by rounding off the optimum values of the variables to the nearest integer values. But, it is generally inaccurate to obtain an integer solution by rounding off in this manner, for there is no guarantee that the deviation from the 'exact' integer solution will not be too large to retain the feasibility.

The linear programming problem with the additional requirement that the variable can take on only, integer values may have the following mathematical form

$$Maximize\ or\ Miminimize\ z = c_1x_1 + c_2x_2 + --- + c_nx_n$$

$$S.t.c \quad a_{i1}x_1 + a_{i2}x_2 + ----- + a_{in}x_n = b_i \quad i = 1,2,\ldots\ldots..m$$

$$x_j \geq 0, \quad j = 1,2,\ldots\ldots..,n$$

Where x_j are valued for $j = 1,2,\ldots\ldots.,p \ (p \leq n)$

We do not know exactly where on this line the objective value of optimal solution lies. We denote this optimal value as arbitrarily on the line. This optimal solution value conceptually divides the value line into two parts:

• above the optimal solution value are upper bounds, values which are above the (unknown) optimal solution value .

• below the optimal solution value are lower bounds, values which are below the (unknown) optimal solution value.

4.3 Quadratic Programming

Quadratic Program problems have linear constraints, but the objective function *f* must be quadratic. Thus, the only difference between such a problem and a Linear Program problem is that some of the terms in the objective function involve the square of a variable or the product of two variables. A number of special algorithms based upon the extending Simplex method have been developed for the Quadratic Program with **convex** quadratic objective function (for the minimization problem) [5]. These algorithms have been implemented in many Quadratic Program solvers.

4. Decomposition and solution algorithm

In this, the decomposition methods and corresponding solution algorithms applied to solve the two combinatorial optimisation problems. We first introduce domain independent general decompositions methods and corresponding solution algorithms. These methods include Danzig-Wolfe decomposition and column generation algorithm, variable fixing applied as decomposition method when solving a MIP. We can introduce some ideas of decomposition methods applied in solving a specific application problem, Nurse Scheduling Problems (NRP) s. **5.1 Decomposition in NRPs**

The idea of intelligently breaking up larger problems into smaller, easier to handle sub problems and then dealing with each sub problem in turn has been shown to work well on nurse rostering [52] and on other scheduling/timetabling problems [5]. In [7], constraints are categorised into shift constraints (which considered the number of staff and the skill category required for each shift), and nurse constraints (which considered the workload for each nurse including nurse preferences, consecutive shifts and the intervals between shifts). The nurse constraints were used to produce all feasible shift patterns of the whole scheduling period for each nurse, independently from shift constraints. The best combinations of these shift patterns are found using mathematical programming and meta-heuristics [7].

In [18], all the feasible weekly shift patterns are pre-defined and associated with costs which are related with preferences, requests, and the number of successive days, etc. These shift patterns are then used to construct nurse rosters by employing different heuristic decoders within a genetic algorithm to schedule both shifts and patterns for the best permutations of nurses. In [10], high quality pre-defined schedules are employed to construct cyclic schedules for a group of nurses with the same requirements. Based on these partial cyclic schedules, the rest of the shifts are assigned to the rest of the nurses with different requirements. The problems can thus be seen as being decomposed into cyclic and noncyclical parts.

5. Conclusion

This paper presents a review of optimisation techniques: CP, OR techniques and local Search. CP and Integer Programming are exact optimisation methods to combinatorial optimisation problems. Global constraints, together with their propagation algorithms, serve as building blocks for both the problem modelling and the problem solving. They can be well used to model and solve the complex and large set of constraints presented in real-world combinatorial optimisation problems. The OR techniques, e.g. Linear Programming, can perform optimality reasoning through the solution to the relaxed problem of the original one, and they can also be used to reduce the search space of the problem. The basic problem can be modelled and solved by Linear Programming or Quadratic Programming. These hybrid methods can seek good quality solutions, not necessary the optimal one, in a very limited computational time. At the same time, we can have the knowledge of the quality of this solution.

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