Some Properties of Quasi-normal operators

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Abstract: In this article we will give some properties of quasi-normal operators in Hilbert spaces. Theobject of this paper isto study some properties of quasi-normal operators. If T_1 and T_2 are quasinormaloperators, we shall obtain conditions under which their sum and product are quasinormal.

Keywords: Hilbert space, Quasinormal operators.

1. Introduction

Let us denote by H the complex Hilbert space and with B(H) the space of all bounded linear operators defined in Hilbert space H. Let T be an operator in B(H). The operator T is called qusinormal if $T(T^*T) = (T^*T)T$. Let $T \in B(H)$, $T = U+iV$ where $U = \text{Re } T = \frac{T+T^*}{2}$ and $V = \text{Im } T = \frac{T-T^*}{2i}$ are the real and imaginary parts of T. We shall write $B^2 = TT^*$ and $C^2 = T^*T$ where B and C are non-negative definite.

In this paper we will study some properties of quasinormal operators. Exactly we will give conditions under which an operator T is quasinormal. Also, we shall that of T_1 and T_2 are qusinormal operators, we shall obtain conditions under which their sum and product are quasinormal.

2. Quasi–normal operators

In this section we will show some properties of quasinormal operators in Hilbert space.

Theorem: 2.1

If $T \in B(H)$ is an invertible quasinormal operator, then

1. $T^*(T^*T) = (T^*T)\, T^*$,
2. $T (T^*T) = (T^*T)\, T$,
3. $T^{-1} (T^*T) = (T^*T)\, T^{-1}$.

Theorem: 2.2

If T is quasinormal operator, then the equality $(T(T^*T))^n = T^n (T^*T)^n$ holds true for all $n \in N$.

Proof:

Observe that,

$$((T^*T)\, T)^2 = (T^*T)\, T\, (T^*T)T = T^2 (T^*T)^2$$

$$\begin{aligned}((T^*T)\, T)^3 &= ((T^*T)\, T)^2\ (T^*T)\, T \\ &= T^2 (T^*T)^2 (T^*T)\, T \\ &= T^3 (T^*T)^3\end{aligned}$$

By inductive argument it is obvious that,

$$((T^*T)\, T)^n = T^n (T^*T)^n.$$

Theorem: 2.3

If T is an operator such that

(i) B commutes with U and V.

(ii) $TB^2 = (C^2T)$.

Then T is quasinormal operator.

Proof:

Since $BU = UB$, $BV = VB$ we have $B^2U = UB^2$, $B^2V = VB^2$, then

$$B^2T + B^2T^* = TB^2 + T^*B^2$$

$$B^2T - B^2\, T^* = TB^2 - T^*\, B^2.$$

This gives, $B^2T = TB^2 = C^2T$

$= (T^*T)T$

$T(T^*T) = (T^*T)T$.

Hence T is quasinormal operator.

Theorem: 2.4

Let T be quasinormal operator and $TB^2 = (C^2T)$. Then B commutes with U and V.

Proof:

Since $TB^2 = C^2T$, we have $T(TT^*) = (T^*T)T$.

Hence, $(TT^*)T^* = T^*(T^*T)$. Since T is quasinormal operator, we have

$$B^2U = TT^*\frac{T+T^*}{2}$$
$$= \frac{TT^*T+TT^*T^*}{2}$$
$$= \frac{(T^*T)T+T^*(T^*T)}{2}$$
$$= \frac{T(TT^*)+(T^*T)T^*}{2}$$
$$= \frac{T^2T^*+T^*TT^*}{2}$$
$$= \frac{T+T^*}{2}TT^*$$
$$= UB^{2.}$$

Since B is non negative definite, it follows that BU = UB. Similarly, BV = VB.

Theorem: 2.5

If T is an operator such that $C^2U = UC^2$, $C^2V = VC^2$. Then T is quasinormal operator.

Proof:

Since If T is an operator such that $C^2U = UC^2$, $C^2V = VC^2$

Then we have,

$C^2(U + iV) = (U + iV)C^2$ and $C^2T = TC^2$, $(T^*T)T = T(T^*T)$.

Hence T is quasinormal operator.

Theorem: 2.6

Let T be quasinormal operator and $B^2T = (C^2T)$. Then

(i) $C^2U = UC^2$

(ii) $C^2V = VC^2$

Proof:

(i) Since $B^2T = C^2T$

$\Rightarrow (TT^*)T = (T^*T)T$

$\Rightarrow T^*(TT^*) = T^*(T^*T)$

Since T is quasi normal operator, we have

$$C^2U = T^*T\frac{T+T^*}{2}$$
$$= \frac{T^*T^2+T^*TT^*}{2}$$
$$= \frac{TT^*T+T^{*2}T}{2}$$
$$= \frac{T+T^*}{2}T^*T$$
$$= UC^2$$

(ii) Similarly,

$C^2V = VC^2$

Theorem: 2.7

Let T_1 and T_2 be two qusinormal operators suchtaht $T_1T_2 = T_2T_1 = T_1^*T_2 = T_2^*T_1 = 0$. Then their sum $T_1 + T_2$ is quasinormal operator.

Proof:

$(T_1 + T_2)[(T_1 + T_2)^*(T_1 + T_2)] = (T_1 + T_2)[(T_1^* + T_2^*)(T_1 + T_2)]$

$= (T_1 + T_2)[T_1^*T_1 + T_1^*T_2 + T_2^*T_1 + T_2^*T_2]$

$= (T_1 + T_2)[T_1^*T_1 + T_2^*T_2]$, since $T_1^*T_2 = T_2^*T_1 = 0$

$= T_1(T_1^*T_1) + T_1(T_2^*T_2) + T_2(T_1^*T_1) + T_2(T_2^*T_2)$

$= T_1(T_1^*T_1) + T_2(T_2^*T_2)$, since T_1, T_2 are Quasinormal operators.

$= (T_1^*T_1)T_1 + (T_2^*T_2)T_2$

$= [(T_1 + T2 *T1+T2](T_1 + T_2)$

Hence $T_1 + T_2$ is quasinormal operator.

Theorem: 2.8

Let T_1 be quasinormal operator and T_2 qusinormal operator. Then their product T_1T_2 is quasinormal operator if the following conditions are satisfied.

(i) $T_1T_2 = T_2T_1$

(ii) $T_1T_2^* = T_2^*T_1$

Proof:

$(T_1T_2)[(T_1T_2)^*(T_1T_2)] = (T_1T_2)[(T_2^*T_1^*)(T_1T_2) = (T_1T_2)(T_1^*T_2^*)(T_1T_2) = T_1(T_2T_1^*)(T_2^*T_1)T_2 = T_1(T_1^*T_2)(T_1T_2^*)T_2 = T_1T_1^*(T_2T_1)$

$(T_2^* T_2) = T_1 T_1^* (T_1 T_2) (T_2^* T_2) = (T_1^* T_1^2)$

$(T_2^* T_2^2) = T_1^* T_2^* (T_1^2 T_2^2) = (T_1^* T_2^*)$

$(T_1 T_2)^2 = (T_2^* T_1^*) (T_1 T_2)^2 = (T_1 T_2)^*$

$(T_1 T_2)^2 = [(T_1 T_2)^* (T_1 T_2)] (T_1 T_2)$
$(T_1 T_2) [(T_1 T_2)^* (T_1 T_2)] = [(T_1 T_2)^* (T_1 T_2)]$
$(T_1 T_2)$

Hence $T_1 T_2$ is quasi normal operators.

References

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