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Finite Volume Methods for Non-Linear Equations

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Abstract:
We consider the classical numerical-type of models. In the first part we deal with finite difference numerical model for irotational water waves. Finite volume methods are based on the integral form. A numerical model for solving the two-dimensional equations is presented. The standard Galerkin method with mixed interpolation is applied. In the second part we consider the water wave equation with a logarithmic nonlinearity. Using the Galerkin method, we establish the existence of solutions of the problem.

Keywords: numerical model, logarithmic nonlinearity, Galerkin method, global solutions, water equation

Introduction

We deal with water wave equation. We want to obtain a model for non-linear equations. In the first part using the fully non-linear model for irrotational water waves in the form (see [1], [2]) given as

$$0 = \delta L = \delta \iint L \, dx \, dt \qquad (1.1)$$

we consider the finite volume method. Finite volume methods are based on the integral form of the conservation law

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx + f\,(q(x_2,t)) - f(q(x_1,t)) = 0 \qquad (1.2)$$

Dingemans (1997) describes several methods with positive-definite Hamiltonian, but these methods are quite tedious and have certain ambiguities regarding the order of certain operators, (see [3], [4]). The present method leads to a positive-definite Hamiltonian and can be fully non-linear if desired. The present model is an additional elliptic equation in the horizontal plane has to be solved (see [6]). High-order non-linear models solve free-surface evolution equations derived from a Hamiltonian under the constraint that the Laplace equation is satisfied exactly in the interior of the fluid domain (see [7]).

In the second part we deal with the existence and decay of solutions of the following problem

$$u_{tt} + Au + u + h(u_t) = kuln|u| \qquad (1.3)$$

with boundary conditions

$$u(x,t) = \frac{\partial u}{\partial v}(x,t) = 0\,, \quad x \epsilon\ \partial\Omega, \ \ t > 0$$
$$u(x,0) = u_0(x)\ ; \qquad u_t(x,0) = u_1(x)$$

where $\Omega \subset R^n$, $n \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$, $k \geq 1$ and $A = (-\nabla)^m$, $(m \geq 1)$, v is the unit outer normal to $\partial\Omega$ and k is a positive real number. This type of problems has applications in many branches of physics such as nuclear physics, optics and geophysics [5,6,11]. In [**8**], Cazenave and Haraux considered

$$u_{tt} - \nabla u = uln|u|^k \qquad (1.4)$$

M. Al-Gharabli And S. A. Messaoudi J. Evol. Equ. and established the existence and uniqueness of the solution for the Cauchy problem. Hiramatsu et al. [9] introduced the following equation

$$utt - u + u + ut + |u|2u = u\,ln\,|u| \qquad (1.5)$$

to study the dynamics, Q-ball in theoretical physics.

1. Finite difference numerical model for irrotational water waves

Fully non-linear model for irrotational water waves in the form (see [1], [2]) given as

$$0 = \delta L = \delta \iint L \, dx \, dt \qquad (1.6)$$

where the Lagrangian density is

$$L(\zeta, \partial t\zeta, \phi, \partial_x \varphi, \partial_z \varphi;\ x, t).$$

$$L = \phi\, \partial_t \zeta - H \quad \text{with} \quad \phi = [\varphi]_z = \zeta$$

where $\zeta(x,t)$ is the surface elevation, $\varphi(x,z,t)$ is the velocity potential. Than the energy density $H(\zeta, \partial_x\varphi, \partial_z\varphi;\ x, t)$ is given by the sum of kinetic and potential energy densities as follows

$$H = \int_{-h}^{\zeta} \frac{1}{2}[(\partial_x\varphi)^2 + (\partial_z\varphi)^2]dz + \frac{1}{2}g\zeta^2$$

while the mass density ρ is taken to be constant and equal to one. Further $h(x)$ is the still-water depth and g is the gravitational acceleration.
Note that the Hamiltonian $\overline{H}(\zeta, \partial_x\varphi, \partial_z\varphi)$ itself is the partial integral of H:

$$\overline{H} = \int H \, dx \qquad (1.7)$$

Let we see the potential $\varphi(x, z, t)$, corresponding with a parabolic behaviour over depth with $\partial_z\varphi = 0$ at the bed and $\varphi = \phi$ at the free surface:

$$\varphi(x,z,t) = \phi(x,t) + f(z;\ h,\zeta)\,\psi(x,t),$$

$$f(z;\ h,\zeta) = \tfrac{1}{2}(z - \zeta)\frac{h+z+\zeta}{h+\zeta} \qquad (1.8)$$

We want time derivatives of $\zeta(x,t)$ and $\phi(x,t)$ to appear in the Euler-Lagrange equations. Note that for a horizontal bottom we have $\partial_z\varphi = 0$ at $z = -h$. The velocity components become:

$$\partial_x\varphi = \partial_z\varphi - \tfrac{1}{2}\left(1 + (\tfrac{h+z}{h+\zeta})^2\right]\psi\partial_x\zeta + f(z;\ h,\zeta)\partial_x\psi \qquad (1.9)$$

where $\partial_z\varphi = \frac{h+z}{h+\zeta}\psi$. Note that $\psi(x,t)$ is the vertical velocity $\partial_z\varphi$ at $z = \zeta(x,t)$.

Energy density H is:

$$H = \tfrac{1}{2}(h+\zeta)\left[\partial_x\varphi - \tfrac{1}{2}\psi\partial_x\zeta - \tfrac{1}{3}(h+\zeta)\partial_x\psi\right]^2 + \tfrac{1}{90}(h+\zeta)[\psi\partial_x\zeta - (h+\zeta)\partial_x\psi]^2 + \tfrac{1}{6}(h+\zeta)\psi^2 + \tfrac{1}{2}g + \zeta^2 \qquad (1.10)$$

We take variations of L with respect to ϕ, ζ and ψ we get from $\delta L = 0$ and introduce $u \equiv \partial_x\varphi$, and note that the discharge $q(x,t)$ and depth-averaged velocity $U(x,t)$ are: $q \equiv (h+\zeta)\,U$, and

$$U = u - \tfrac{2}{3}\psi\,\partial x\zeta - \tfrac{1}{3}(h+\zeta)\partial_x\psi \qquad (1.11)$$

Step by step following all actions we have to solve two time-evolution equations for $\zeta(x,t)$ and $u(x, t)$, as well as an elliptic equation for $\psi(x,t)$. For full steps we can [4]. Then the system of equations to be solved can be written as:

$$\partial_t\zeta + \partial_x((h+\zeta)\,U) = 0$$

Finally,

$$(h+\zeta]\psi\left[\tfrac{1}{3} + \tfrac{7}{15}(\partial_x\zeta)^2\right]^2 - \left[\tfrac{2}{3}(h+\zeta)u - \tfrac{1}{5}(h+\zeta)^2\partial_x\psi\right]\partial_x\zeta + \partial_x\left[\tfrac{1}{3}(h+\zeta)^2 u - \tfrac{1}{5}(h+\zeta)^2\psi\,\partial_x\zeta - \tfrac{2}{15}(h+\zeta)^3\partial_x\psi\right] = 0 \qquad (1.12)$$

2. Preliminaries

In this section we deal with the existence of solutions of the following problem for the water wave equation with logarithmic term.

$$u_{tt} + Au + u + h(u_t) = kuln|u| \qquad (2.1)$$

with boundary conditions

$$u(x,t) = \frac{\partial u}{\partial v}(x,t) = 0, \quad x\epsilon\,\partial\Omega, \ t > 0$$
$$u(x,0) = u_0(x); \qquad u_t(x,0) = u_1(x)$$

where $\Omega \subset R^n$, $n \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$, $k \geq 1$ and $A = (-\Delta)^m$, $(m \geq 1)$, v is the unit outer normal to $\partial\Omega$ and k is a positive real number, $x\epsilon\,\Omega$, $t > 0$.

Definition 2.1. (*weak solution of eq.* (2.1))

A continuous function $u = u(t,x)$ is a global weak solution to the Cauchy problem (1.2) if:

$u = u(t,x) \in C((0,\infty) \times \Omega) \cap L^{\infty}(R, H^m(\Omega))$ and $\|u\|_{H^m(\Omega)} \le \|u_0\|_{H^m(\Omega)} \quad \forall\, t > 0$ $u(t,x)$ satisfies equation (1-2) in the sense of distributions.

Lemma 2.2.. Logarithmic Sobolev inequality

(see [13,14]). Let $\boldsymbol{u}$ be any function in $H_0{}^m(\Omega)$ and $a > 0$ be any number. Then

$$2\int_{\Omega} |u|^2 \ln|u|\, dx \le \frac{1}{2}\, \|u\|^2 ln\|u\|^2 + \frac{ca^2}{2\pi}\, \|Au\|^2 - (1 + lna)\|u\|^2 \qquad (2.2)$$

Lemma 2.3. Logarithmic Gronwall inequality

(see [8]). Let $\boldsymbol{c} > 0$ and $\gamma \in (0, T, \Omega)$. Let ω be any function ω: $[0, T[\ \to [1,\infty[$ satisfies

$$\omega \le c\,(1 + \int_0^t \gamma(s)\,\omega(s) \ln\omega(s)\, ds)\ , 0\le t \le T$$

then $\omega \le c \exp\ (c \int_0^t \gamma(s)\, ds)\ ,\ 0\le t \le T$ (2.3)

Lemma 2.4. The Cautchy – Schwartz inequlity

Recall: For the Hilbert space with a norm (u,v) and its resulted norm $\|(u,v)\| = \sqrt{(u,v)}$, than the Cauchy-Schwartz inequality is the following, $|u(x), v(x)| \le \|u\|\|v\|$

3. Galerkin method for existence of solutions

We use the standard Faedo–Galerkin method for the existence of solutions for the water wave equation with logarithmic term (2.1).

Theorem 3.1

Let $(u_0, u_1) \in H_0^m(\Omega) \times L^2(\Omega)$. Then, problem of equations (2.1) has a global week solution as $u = u(t,x) \in C((0,T)$, $H_0^m(\Omega) \cap C^1(0,T)$, $L^2(\Omega) \cap C^2(0,T)$, $H^m(\Omega))$

Proof: To proof the theorem we consider the standard Faedo-Galerkin method. We take an orthogonal basis of the space $H_0^m(\Omega)$ in the form $\{\omega_j\}_{j=1}^{\infty}$. This is othonormal in $L^2(\Omega)$. Let $V_m = span\{\omega_{1,}\omega_{2,} \ldots \ldots \omega_m\}$ and let the projections of the initial data on the subspace V_m be given by

$$u_0^m(x) = \sum_{j=1}^m a_j\omega_j(x)\ ,\ u_1^m(x) = \sum_{j=1}^m b_j\omega_j(x)$$

where $u_0^m \to u_0$ in $H_0^m(\Omega)$ and $u_1^m \to u$ in $L^2(\Omega)$, as $m \to \infty$.

We search for an approximate solution $u^m(x,t) = \sum_{j=1}^m g_j^m(t)\omega_j(x)$ of the approximate problem in V_m

$$\begin{cases} \int_\Omega (u_{II}^m \mathrm{w} + \Delta u^m \Delta \mathrm{w} + u^m \mathrm{w} + h(u_t^m)\mathrm{w})dx = \int_\Omega \mathrm{w}u \\ u^m(0) = u_0^m = \sum_{j=1}^m (u_{0,}\mathrm{w}_j)\,\mathrm{w}_j \\ u_1^m(0) = u_1^m \sum_{j=1}^m (u_{1,}\mathrm{w}_j)\,\mathrm{w}_j \end{cases} \qquad (3,4)$$

This leads to a system of ODE$_s$ for unknown functions $g_j^m(t)$. Based on standard excistence theory for ODE, one can obtain functions:

$$g_j: [0, t_m) \to R, \qquad j = 1, 2, \ldots, m,$$

which satisfy (3,4) in a maximal interval $[0, t_m), t_m \in (0, T\,]$. Next we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let w=u_t^m in (3,4) and integrate by parts to obtain

$$\frac{d}{dt}E^m(t) = -\int_\Omega u_t^m h(u_t^m)dx \le 0 \qquad (3.5)$$

where,

$$E^m(t) = \frac{1}{2}\left(\|u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \left(\frac{k+2}{2}\right)\|u^m\|_2^2 - \int_\Omega |u^m|^2 ln|u^m|^k dx\right) \qquad (3.6)$$

The last inequality together with the Logarithmic Sobolev inequality leads to

$$\|u_t^m\|_2^2 + \left(1 - \frac{ka^2c_p}{2\pi}\right)\|\Delta u^m\|_2^2 + \left[\left(\frac{k+2}{2}\right) + k(1 + \ln a\right]\|u^m\|_2^2 \le C + \|u_t^m\|_2^2 \ln\|u^m\|_2^2 \qquad (3.7)$$

Choossing $e^{\left(-\frac{3}{2}-\frac{1}{2k}\right)} < a < \sqrt{\frac{2\pi}{kcp}}$ will make $1-\frac{ka^2cp}{2\pi} > 0$ and $\left(\frac{k+2}{2}\right)+k(1+\ln a) \geq 0$

This selection is possible thanks to (A2). So, we get

$$\|u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \leq C(1 + \|u^m\|_2^2 \ln\|u^m\|_2^2) \quad (3.8)$$

Note $u^m(\cdot,t) = u^m(\cdot,0) + \int_0^t \frac{\partial u^m}{\partial_s}(\cdot,s)ds$

Then, using Cauchy-Schwarz' inequality, we get

$$\|u^m(t)\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2\left\|\int_0^t \frac{\partial u^m}{\partial_s}(\,s)ds\right\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2T\int_0^t \|u_t^m(s)\|_2^2 ds \quad (3.9)$$

$$\|u^m(t)\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2TC\left(1 + \int_0^t \|u^m\|_2^2 \ln\|u^m\|_2^2\, ds\right) \quad (3.10)$$

If we put $C_1 = \max\{2TC, 2\|u^m(0)\|_2^2\}$, (3.10) leads

$$\|u^m\|_2^2 \leq 2C_1\left(1 + \int_0^t (C_1 + \|u^m\|_2^2)\ \ln\ (C_1 + \|u^m\|_2^2)\, ds\right)$$

Applying the Logarithmic Gronwall inequality to the last inequality, we obtain the following estimate

$$\|u^m\|_2^2 \leq 2C_1 e^{2C_1T} \leq 2C_2$$

Hence, from the inequality (3.8) it follows that:

$$\|u_t^m\|_{L^2(\Omega)}^2 + \|\Delta u^m\|_{L^2(\Omega}^2 + \|u^m\|_{L^2(\Omega}^2 \leq C_3$$

where C_3 is a positive constant independent of m and t. This implies

$$\sup_{t\epsilon(0,t_m)} \|u_t^m\|_{L^2(\Omega)}^2 + \sup_{t\epsilon(0,t_m)} \|\Delta u^m\|_{L^2(\Omega)}^2 + \sup_{t\epsilon(0,t_m)} \|u^m\|_{L^2(\Omega)}^2 \leq C_4 \quad (3.11)$$

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain, from (3.11),

$$\begin{cases} u^m \text{ is uniformly bounded in } L^\infty(0,T;\ H_0^m(\Omega)) \\ u_t^m \text{ is uniformly bounded in } L^\infty(0,T;\ L^2(\Omega)) \end{cases} \quad (3.12)$$

which implies that there exists a subsequence of u^m (still denoted by u^m), such that

$$\begin{cases} u^m \rightharpoonup u \quad \text{weakly}^* \quad \text{in } L^\infty(0,T;\ H_0^m(\Omega)) \\ u_t^m \rightharpoonup u_t \quad \text{weakly}^* \quad \text{in } L^\infty(0,T;\ L^2(\Omega)) \\ u^m \rightharpoonup u \quad \text{wekaly} \quad \text{in } L^2(0,T;\ H_0^m(\Omega)) \\ u_t^m \rightharpoonup u_t \quad \text{weakly} \quad \text{in } L^2(0,T;\ L^2(\Omega)) \end{cases} \quad (3.13)$$

Making use of Aubin –Lions' theorem, we find, up to a subsequence, that $u^m \to u$ strongly in $L^2(0,T;\ L^2(\Omega)$ and $u^m \to u$ a.e in $\Omega \times (0,T)$.

Since the map $s \to s\ln|s|^k$ is continuous, we have the convergence $u^m \ln|u^m|^k \to u\ln|u|^k$ in $\Omega \times (0,T)$

Using the embedding of $H_0^m(\Omega)$ in $L^\infty(\Omega)$ $(\Omega \subset R^2)$, it is clear that $u^m \ln|u^m|^k$ is bounded in L^∞ $(\Omega \times (0,T))$. Next, taking into account the Lebesgue bounded convergence theorem (Ω is bounded), we get converge strongly

$$u^m \ln|u^m|^k \to u\ln|u|^k \quad \text{in} \quad L^2\big(0,T;\ L^2(\Omega)\big) \quad (3.14)$$

Next, we prove that $h(u_t^m)$ is bounded in $L^2(0,T;\ L^2(\Omega)$. For this purpose, we consider two cases:

Case 1. H is linear on $[0,\varepsilon]$. Then using (2.1) and Young' s inequality, we get

$$\int_\Omega h^2(u_t^m)dx \le c\int_\Omega u_t^m h(u_t^m)dx - \int_\Omega |u_t^m|^2 dx \le \frac{c}{4\delta_0}\int_\Omega |u_t^m|^2 dx + \delta_0\int_\Omega h^2(u_t^m)dx \qquad (3.15)$$

for a suitable choice of δ_0 , and using the fact that u_t^m is bounded in $L^2\big((0,T),L^2(\Omega)\big)$, we obtain

$$\int_0^T\int_\Omega h^2(u_t^m)dxdt \le c \qquad (3.16)$$

Case 2. Let $0<\varepsilon_1\le\varepsilon$ such that

$$sh(s)\le \min\{\varepsilon, H(\varepsilon)\} \text{ for all } |s|\le\varepsilon_1 \qquad (3.17)$$

$$\begin{cases} s^2+h^2(s)\le H^{-1}\big(sh(s)\big) & \text{for all } |s|\le\varepsilon_1 \\ c_1'|s|\le|h(s)|\le c_2'|s| & \text{for all } |s|\ge\varepsilon_1 \end{cases} \qquad (3.18)$$

Define the following sets

$$\Omega_1=\{x\in\Omega : |u_t^m|\le\varepsilon_1\},\ \Omega_2=\{x\in\Omega : |u_t^m|\le\varepsilon_1\} \qquad (3.19)$$

Then, using (5.7) and (3.19) leads to

$$\int_\Omega h^2(u_t^m)dx = \le c_2'\int_{\Omega_2}|u_t^m|^2dx + \int_{\Omega_1}\big(|u^m|_t^2 + h^2(u_t^m)\big)dx - \int_{\Omega_1}|u_t^m|^2dx \le c_2'\int_{\Omega_2}|u_t^m|^2dx + \int_{\Omega_1} H^{-1}\,(u_t^m h(u_t^m))dx \qquad (3.20)$$

$$Let \qquad J^m(t) := \int_{\Omega_1} u_t^m h(u_t^m)dx$$

Using (3.20) and Jensen's inequality, we obtain

$$\int_\Omega h^2(u_t^m)dx \le c\int_\Omega |u_t^m|^2dx + H^{-1}\big(J(t)\big) = c\int_\Omega|u_t^m|^2dx + \frac{H'\left(\varepsilon_0\frac{E^m(t)}{E^m(0)}\right)}{H'\left(\varepsilon_0\frac{E^m(t)}{E^m(0)}\right)}H^{-1}\big(J(t)\big) \qquad (3.21)$$

Using the convexity of $H (H'$ is increasing), we obtain for $t\in(0,T)$,

$$H'\left(\varepsilon_0\frac{E^m(t)}{E^m(0)}\right)\ge H'\left(\varepsilon_0\frac{E^m(T)}{E^m(0)}\right)=c$$

Let H^* be the convex conjugate of H in the sense of Young , then, for $s\in(0,H'(r^2)]$

$$H^*(s)=s(H')^{-1}(s)-H[(H')^{-1}(s)] \le s(H')^{-1}(s) \qquad (3.22)$$

Using the general Young inequality $AB\le H^*(A)+H(B)$, if $A\in(0,H'(r^2)]$, $B\in(0,r^2]$

For $A=H'\left(\varepsilon_0\frac{E^m(t)}{E^m(0)}\right)$ and $B=H^{-1}\big(J^m(t)\big)$

and using the fact that $E^m(t)\le E^m(0)$, we get

$$\int_\Omega h^2(u_t^m)dx \le c\varepsilon_0\frac{E^m(t)}{E^m(0)}H'\left(\varepsilon_0\frac{E^m(t)}{E^m(0)}\right) - C(E^m)'(t) \le c\int_\Omega|u_t^m|^2dx + c \le -C(E^m)'(t) \qquad (3.23)$$

Integrating (3.23) over $(0,T)$, we obtain

$$\int_0^T\int_\Omega h^2(u_t^m)dxdt \le c\int_0^T|u_t^m|^2dxdt + cT - C(E^m(T) - E^m(0) \qquad (3.24)$$

Using (3.5) and the fact that u_t^m is bounded in $L^2\big((0,T),L^2(\Omega)\big)$, we conclude that $h(u_t^m)$ is bounded in $L^2\big((0,T),L^2(\Omega)\big)$. So we find, up to a subsequence that.

$$h(u_t^m) \rightharpoonup \chi \text{ in } L^2\big((0,T),L^2(\Omega)\big) \qquad (3.25)$$

Now, we integrate (3.4) over $(0,t)$ to obtain

$$\int_\Omega u_t^m \mathrm{w}dx - \int_\Omega u_1^m \mathrm{w}dx + \int_0^t\int_\Omega \Delta u^m(s)\Delta \mathrm{w}dxds + \int_0^t\int_\Omega u^m(s)\mathrm{w}dxds + \int_\Omega h(u_t^m)\mathrm{w}dxds = \int_\Omega \int_0^t \mathrm{w}u^m(s)\ln|u^m(s)|^k dxds, \ \ \forall \mathrm{w} \in V_m \ (3.26)$$

Convergences (3.3), (3.13), (3.14) and (3.25) are sufficient to pass the limit in (3.26) as $m \to \infty$, get

$$\int_\Omega u_t \mathrm{w}dx = \int_\Omega u_1\mathrm{w}dx - \int_0^t\int_\Omega \Delta u(s)\Delta\mathrm{w}dxds - \int_0^t\int_\Omega u(s)\mathrm{w}dxds - \int_0^t\int_\Omega \chi(s)\mathrm{w}dxds \int_\Omega \int_0^t u(s)\,\mathrm{w}\ln|u(s)|^k dx, \quad (3.27)$$

which implies that (3.27) is valid for any $\mathrm{w}\in H_0^m(\Omega)$. Using the fact that the terms in the right-hand side of (3.27) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$; hence, it is differentiable for a.e. $t \in R^+$. Thus, differentiating (3.27), we obtain for a.e. $t \in (0,T)$.

$$\int_\Omega u_{tt}(x,t)\mathrm{w}(x)dx + \int_\Omega \Delta u(x,t)\Delta\mathrm{w}(x)dx + \int_\Omega u(x,t)\mathrm{w}(x)dx + \int_\Omega \chi(\,t)\mathrm{w}(x)dx = \int_\Omega \mathrm{w}(x)u(x,t)\ln|u(x,t)|^k dx \qquad (3.28)$$

On the other hand, since h is a no decreasing monotone function, one has

$$X^m := \int_0^T\int_\Omega (u_t^m - v)\big(h(u_t^m)-h(v)\big)dxdt \geq 0, \ \ v \in L^2\big(0,T;\ L^2(\Omega)\big) \qquad (3.29)$$

Now, integrate (3.6) over $(0,t)$ and taking $m \to \infty$, we obtain

$$0 \leq \limsup X^m \leq \|u_1\|_2^2 + \|\Delta u_0\|_2^2 + \left(\frac{k+2}{4}\right)\|u_0\|_2^2 - \int_\Omega |u_0|\ln|u_0|dx - \left(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \left(\frac{k+2}{4}\right)\|u\|_2^2 - \int_\Omega |u|\ln|u|dx\right) - \int_0^t\int_\Omega \chi(t)vdxds - \int_0^t\int_\Omega (u_t - v)h(v)dxds \qquad (3.30)$$

Replacing w by u_t in (3.28) and integrating over $(0,T)$, to obtain

$$\limsup X^m \leq \|u_1\|_2^2 + \|\Delta u_0\|_2^2 + \left(\frac{k+2}{4}\right)\|u_0\|_2^2 - \int_\Omega |u_0|\ln|u_0|dx - \left(\|u_t\|_2^2 + \|\Delta u\|_2^2 + \left(\frac{k+2}{4}\right)\|u\|_2^2 - \int_\Omega |u|\ln|u|dx\right) - \int_0^t\int_\Omega \chi(t)vdxds \qquad (3.31)$$

Combining (3.30) with (3.31)

$$0 \leq \limsup X^m \leq \int_0^t \int_\Omega \chi(t) u_t dxds - \int_0^t \int_\Omega \chi(t) v dxds - \int_0^t \int_\Omega h(v)(u_t - v) v dxds \leq \int_0^t \int_\Omega \big(\chi(t) - h(v)\big)(u_t - v) dxds \qquad (3.32)$$

Hence,

Let $v = \lambda\psi + u_t$, $\psi \in L^2\big((0,T), L^2(\Omega)\big)$. So, we get ,

$$-\lambda \int_0^t \int_\Omega \big(\chi(t) - h(\lambda\psi + u_t)\big)\psi dxds \leq 0, \quad \forall\psi \in L^2\big((0,T), L^2(\Omega)\big).$$

$\int_0^t \int_\Omega \big(\chi(t) - h(\lambda\psi + u_t)\big)\psi dxds \leq 0, \quad \forall\psi \in L^2\big((0,T), L^2(\Omega)\big)$. As $\lambda \to 0$, we have

$$\int_0^t \int_\Omega \big(\chi(t) - h(u_t)\big)\psi dxds \leq 0, \quad \forall\psi \in L^2\big((0,T), L^2(\Omega)\big). \qquad (3.33)$$

Similarly, for $\lambda < 0$, we get

$$\int_0^t \int_\Omega \big(\chi(t) - h(u_t)\big)\psi dxds \geq 0, \quad \forall\psi \in L^2\big((0,T), L^2(\Omega)\big). \qquad (3.34)$$

Thus, (3.31) and (3.33) imply that $\chi = h(u_t)$. Hence (3.28) becomes

$$\int_\Omega u_{tt}(x,t)\text{w}(x)dx + \int_\Omega \Delta u(x,t)\Delta\text{w}(x)dx + \int_\Omega u(x,t)\text{w}(x)dx + \int_\Omega h(u_t)\text{w}(x)dx = \int_\Omega \text{w}(x)u(x,t)\ln|u(x,t)|^k dx \ , \forall\text{w} \in H_0^m(\Omega) \qquad (3.35)$$

$u^m \rightharpoonup u$ wekaly in $L^2(0,T;\ H_0^m(\Omega))$

$u_t^m \rightharpoonup u_t$ weakly in $L^2(0,T;\ L^2(\Omega))$ (3.36)

Thus, using Lion' s Lemma [30], we obtain

$$u^m \to u \quad \text{in } C([0,T], L^2(\Omega)) \qquad (3.37)$$

Therefore, $u^m(x,0)$ makes sense and $u^m(x,0) \to u(x,0)$ in $L^2(\Omega)$

Also, we have $u^m(x,0) = u_0^m(x) \to u_0(x)$ in $H_0^m(\Omega)$

Hence, $u(x,0) = u_0(x)$

Now, multiply (3.4) by $\emptyset \in C_0^\infty(0,T)$ and integrate over $(0,T)$, we obtain for any $\text{w} \in V_m$

$$-\int_0^T \int_\Omega u_t^m(t)\text{w}\emptyset'(t)dxdt = -\int_0^T \int_\Omega \Delta u^m(t)\Delta\text{w}\emptyset(t)dxdt - \int_0^T \int_\Omega u^m \text{w}\emptyset(t)dxdt - \int_0^T \int_\Omega u_t^m \text{w}\emptyset(t)dxdt + \int_0^T \int_\Omega \text{w}u_m \ln|u_m|^k \emptyset(t)dxdt \qquad (3.38)$$

As $m \to \infty$, we have for any $\text{w} \in H_0^m(\Omega)$ and any $\emptyset \in C_0^\infty(0,T)$

$$-\int_0^T \int_\Omega u_t(t)\text{w}\emptyset'(t)dxdt = -\int_0^T \int_\Omega \Delta u(t)\Delta\text{w}\emptyset(t)dxdt$$

$$-\int_0^T \int_\Omega u w \emptyset(t) dxdt$$

$$-\int_0^T \int_\Omega u_t w \emptyset(t) dxdt$$

$$+-\int_0^T \int_\Omega w \emptyset(t) u \ln|u|^k \, dxdt \qquad (3.39)$$

This means (see [32])

$$u_{tt} \in L^2([0,T), H^{-m}(\Omega))$$

Recalling that $u_t \in L^2(0,T;\ H_0^m(\Omega))$, we obtain

$$u_t \in C([0,T),)H^{-m}(\Omega)$$

So, $u_t^m(x,0)$ makes sense and

$$u_t^m(x,0) \to u_t(x,0) \ \text{ in } \ H^{-m}(\Omega)$$

But

$$u_t^m(x,0) = u_1^m(x) \to u_1(x) \ \text{ in } \ L^2(\Omega)$$

Hence, $\qquad u_t(x,0) = u_1(x).$

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