

Support Neighbourly Irregular Graphs

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Abstract

A new family of irregular graphs namely support neighbourly irregular graph has been introduced and studied for its properties in this paper. In any graph, the support of a vertex is the sum of degrees of its neighbours. A connected graph G is said to be support neighbourly irregular (or simply SNI), if no two adjacent vertices in G have same support. A necessary and sufficient condition for a graph to be SNI has been established and the relationship of SNI graphs with other family irregular graphs have been discussed in this paper.

Keywords: Irregular graphs, Highly irregular graphs, Neighbourly irregular graphs, Pairable vertices, Neighbourhood Highly Irregular graphs, Support neighbourly irregular graphs.

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1 Introduction

Throughout this paper, by a graph we mean a finite, simple, connected, undirected graph. Notations and terminology are as in [10]. In a graph $G(V, E)$, for any vertex $v \in V$, the *open neighbourhood* of v is the set of all vertices adjacent to v . That is, $N(v) = \{u \in V(G) / uv \in E(G)\}$. The degree of v is denoted by $d(v)$. The *closed neighbourhood* of v is defined by $N[v] = N(v) \cup \{v\}$. Clearly, if $N[u] = N[v]$, then u and v are adjacent and $d(u) = d(v)$. A *full vertex* of G is a vertex which is adjacent to every other vertices of G . Two vertices u and v are said to be *pairable vertices* if $N[u] = N[v]$. A vertex v is said to be a *k - regular adjacency vertex* (or simply a *k - RA vertex*) if $d(u) = k$ for all $u \in N(v)$. A vertex is called an *RA vertex* if it is a $k - RA$ vertex for some $k \geq 1$.

The concept of support of a vertex has been introduced and studied by Selvam Avadayappan and G. Mahadevan [5]. The *support* $s_G(v)$ or simply $s(v)$ of a vertex v is the sum of degrees of its neighbours. That is, $s(v) = \sum_{u \in N(v)} d(u)$. The study on this parameter has its own importance as any two vertices of same degree need not be of same importance in any graph unless they are isomorphic images of each other. The degrees of its neighbours contribute much in determining the weightage of a vertex in a graph. Hence it becomes essential to study about the degrees of neighbour vertices also.

Regarding the study about irregularity in graphs, there is no clear cut boundary defined so far to classify the non regular graphs into exact classes. Several attempts have been made to group some non regular graphs having similar properties. On that line, the concept of highly irregular graphs has been studied by Yousef Alavi and others in [2].

A connected graph G is said to be a *highly irregular graph* or simply a HI graph, if no two neighbours of every vertex have same degree. In a highly irregular

graph, there will be at least two vertices of maximum degree. Some highly irregular graphs are illustrated in Figure 1. For more results on highly irregular graphs one can refer [1], [2], [3] and [13].

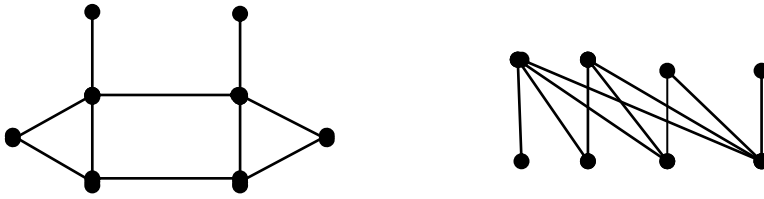


Figure 1

A connected graph G is said to be a *totally segregated graph*, if no two adjacent vertices have the same degree. It has been introduced by Jackson and Roger in [12]. Later it has been studied independently by Gnaana Bhargava and Ayyaswamy in [11]. They called this graph as *neighbourly irregular* graphs.

Some neighbourly irregular graphs are shown in Figure 2. Few results on neighbourly irregular product graphs have been obtained in [8].

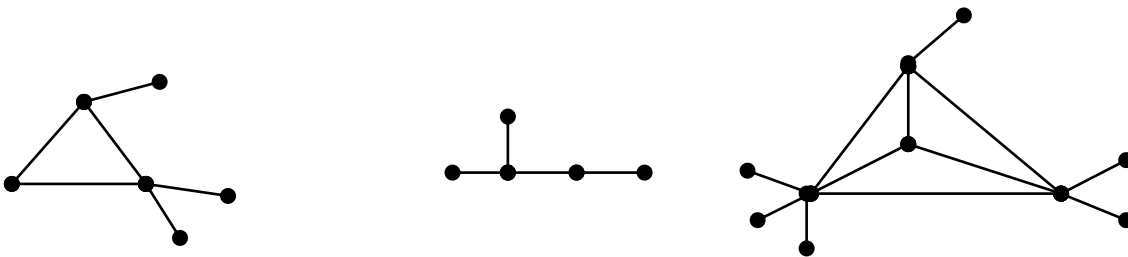


Figure 2

In [14], Swaminathan and Subramanian have introduced a new type of irregular graphs called neighbourhood highly irregular graphs. A connected graph G is said to be *neighbourhood highly irregular* (or simply NHI), if any two distinct vertices in the open neighbourhood of v , have distinct closed neighbourhood sets. Examples of NHI graphs are given in Figure 3. Equivalently, it is observed in [4] that a graph G is NHI if and only if it contains no pairable vertices. Some more results on NHI graphs established in [6], [7] and [9].

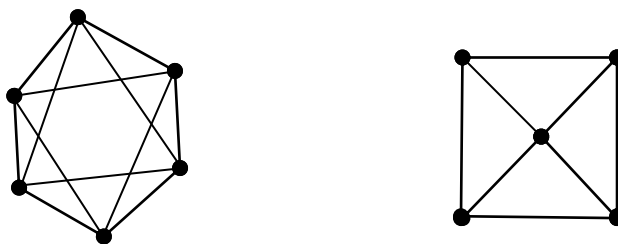
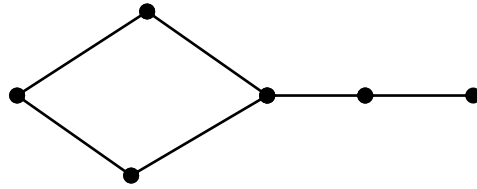


Figure 3

As an addition to this kind of classification, we introduce a new family of irregular graphs namely support neighbourly irregular graphs. A connected graph is said to be *support neighbourly irregular* (or simply SNI), if no two vertices having same support are adjacent. A graph H proving the existence of SNI graphs is shown in Figure 4.



H

Figure 4

In this paper, we study the properties of SNI graphs. Also, some relationships of this family of irregular graphs with that of other irregular graphs have been discussed here.

2 Main Results

The following facts can be easily verified for a SNI graph.

Fact 2.1 Paths P_n , $n \neq 5$, Cycles C_n , $n \geq 3$, Complete graphs K_n , $n \geq 1$, complete bipartite graphs $K_{m,n}$, $m \geq 1$, $n \geq 1$, are not SNI.

Fact 2.2 Not all NI graphs are SNI. In fact the two families are completely independent. For example the graph H shown in Figure 4 is SNI not NI. In Figure 5, the graph G_1 is NI but not SNI, G_2 is both NI and SNI whereas G_3 is neither NI nor SNI.

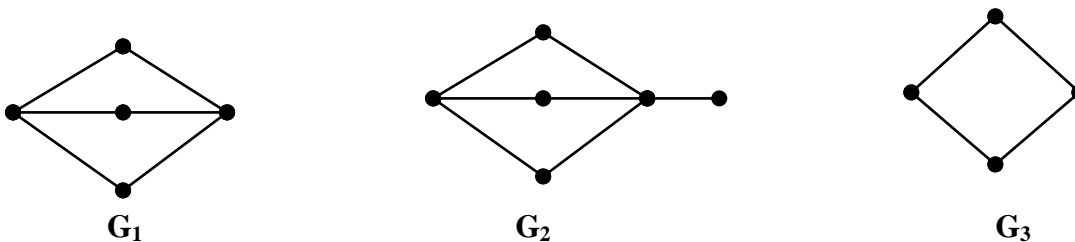


Figure 5

Fact 2.3 Any HI graph is not SNI.

For, let G be a HI graph. Then G contains two adjacent vertices u and v of maximum degree. Then clearly $s(u) = s(v)$ and hence G is not SNI.

Fact 2.4 All SNI graphs are NHI.

For, let G be an SNI graph. If possible suppose that G is not NHI. Then G contains pairable vertices u and v so that $N[u] = N[v]$, that is, $s(u) = s(v)$, which is a contradiction. Hence G is an NHI graph.

Note that the converse of the above result is not true. For example, the cycle C_n , $n \geq 3$ is an NHI graph which is not SNI.

Fact 2.5 Regular graphs are not SNI.

Fact 2.6 Any graph with more than one full vertex is not SNI.

Fact 2.7 The complement of SNI graph need not be SNI. For example, the graph H shown in Figure 4 is SNI but its complement is not SNI.

In a graph G , the subdivision of an edge uv is the process of deleting the edge uv and introducing a new vertex w and the new edges uw and vw . If every edge of G is subdivided exactly once, then the resultant graph is denoted by $S_1(G)$ and is called the subdivision graph of G . For example, the graph $K_{1,5}$ and its subdivision graph $S_1(K_{1,5})$ are given in Figure 6.

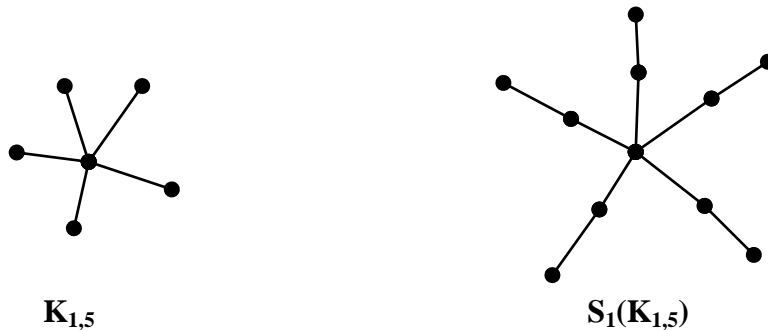


Figure 6

It is clear that there is no SNI tree of order at most 4. But for any $n \geq 5$, we can construct an SNI tree of order n . For example, if n is an odd integer of the form $2m + 1$, $m \geq 2$, then the subdivision graph $S_1(K_{1,m})$ is the required SNI tree of order n . On the other hand, attaching a pendant vertex at the central vertex of $S_1(K_{1,m})$ gives the required SNI graph of even order $2m + 2$, $m \geq 2$. As an illustration an SNI tree of order 12 is shown in Figure 7.

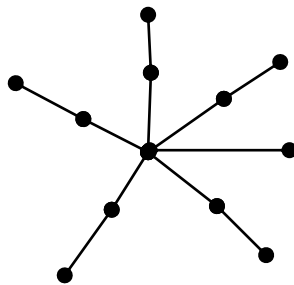


Figure 7

The above construction shows the existence of SNI trees of order n with maximum degree $\lfloor \frac{n}{2} \rfloor$. For any i , $1 \leq i \leq m$, subdivision of i edges in $K_{1,m}$ results in an SNI tree of order n with maximum degree $n - i - 1$. Hence $n - 2$ is the upper bound for the maximum degree in an SNI tree. Also further subdivision of any edge in $S_1(K_{1,m})$ yields an SNI tree of order n in which the maximum degree is less than $\lfloor \frac{n}{2} \rfloor$. And obviously the existence of SNI tree P_5 fixes the lower bound of maximum degree to be 2.

The following theorem proves a necessary and sufficient condition for a graph to be SNI. For any edge $uv \in E(G)$, let $C(uv) = \sum_{x \in N(u) \cap N(v)} d(x)$. It is clear that for any edge uv , $C(uv) + \sum_{x \in N[u] \setminus N[v]} d(x) + d(v) = s(u)$.

Theorem 2.8 A graph G is SNI if and only if $\sum_{x \in N[u] \setminus N[v]} d(x) - \sum_{y \in N[v] \setminus N[u]} d(y) \neq d(u) - d(v)$, for any edge $uv \in E(G)$.

Proof Let G be any SNI graph. Then no two adjacent vertices have same support. If possible, let $\sum_{x \in N[u] \setminus N[v]} d(x) - \sum_{y \in N[v] \setminus N[u]} d(y) = d(u) - d(v)$, for some edge $uv \in E(G)$. That is, $\sum_{x \in N[u] \setminus N[v]} d(x) + d(v) = \sum_{y \in N[v] \setminus N[u]} d(y) + d(u)$. Adding the sum $C(uv)$ on both sides, we get $s(u) = s(v)$, which is a contradiction.

Conversely suppose that $\sum_{x \in N[u] \setminus N[v]} d(x) - \sum_{y \in N[v] \setminus N[u]} d(y) \neq d(u) - d(v)$, for any edge $uv \in E(G)$. That is, $\sum_{x \in N[u] \setminus N[v]} d(x) + d(v) \neq \sum_{y \in N[v] \setminus N[u]} d(y) + d(u)$. This forces that $s(u) \neq s(v)$ for any edge $uv \in E(G)$. Hence G is an SNI graph. ■

Theorem 2.9 Let G be a graph with a full vertex. If G is SNI, then $\sum_{x \in N(v)^c} d(x) + 1 \neq n$, for every non full vertex $v \in V(G)$.

Proof Let G be an SNI graph of order n and $\Delta(G) = n - 1$. Then by Fact 2.6, G contains only one full vertex, say u . Now for any vertex $v \neq u$ in G , we have $uv \in E(G)$ and hence by the above theorem, $\sum_{x \in N[u] \setminus N[v]} d(x) - \sum_{y \in N[v] \setminus N[u]} d(y) \neq d(u) - d(v)$. But u is the full vertex in G . Therefore, $\sum_{x \in N[u] \setminus N[v]} d(x) - \sum_{y \in N[v] \setminus N[u]} d(y) = \sum_{x \in N[v]^c} d(x)$. Thus $\sum_{x \in N[v]^c} d(x) \neq n - 1 - d(v)$ and hence $\sum_{x \in N(v)^c} d(x) + 1 \neq n$ for every non full vertex $v \in V(G)$. ■

Theorem 2.10 The set of all RA vertices in a SNI graph is independent.

Proof Let G be an SNI graph. Let S denote the set of RA vertices in G . If S contains at most one vertex, then there is nothing to prove. Suppose $|S| > 1$. If possible, let u and v be any two vertices in S which are adjacent. Then it is easy to note that $s(u) = s(v) = d(u)d(v)$, which is a contradiction, since G is SNI. Hence we conclude that S is independent. ■

Theorem 2.11 Let G be an SNI graph. Then $G \vee K_m$ is SNI if and only if the following conditions hold:

- (i) $m = 1$,
- (ii) G contains no full vertex and
- (iii) $s(v) - s(u) + d(v) - d(u) \neq 0$, for any edge uv in G , with $d(u) \neq d(v)$.

Proof Let G be an SNI graph. Assume that $G \vee K_m$ is SNI. If $m > 1$ or if G contains a full vertex, then $G \vee K_m$ contains more than one full vertex which is a contradiction, by Fact 2.6. Hence (i) and (ii). Now if possible, let uv be an edge in G such that $s(v) + d(v) = s(u) + d(u)$. Then u and v have same support in $G \vee K_1$, which is impossible.

Conversely assume that all the conditions (i), (ii) and (iii) hold. Clearly by (i) and (ii), $G \vee K_1$ contains exactly one full vertex with unique support and the support of any other vertex in G gets increased by $s(v) + d(v) + |V(G)|$ in $G \vee K_1$. Since G is SNI and also by (iii), we have, support of any two adjacent vertices are distinct. Therefore $G \vee K_1$ is SNI. ■

Theorem 2.12 Every graph of order at least 3 is an induced subgraph of an SNI graph.

Proof Let G be any graph with at least 3 vertices. If G itself is an SNI graph, then there is nothing to prove. Otherwise, G contains an edge uv such that $s(u) = s(v)$. Introduce two new vertices x and y and two new edges xy and ux in G . Let G_1 be the resultant graph. Clearly $s(u) \neq s(v)$ in G_1 . If G_1 is SNI, then we have done with. If not, repeat the procedure for an edge with end vertices of same support in G_1 . Since G is finite,

the procedure terminates after a few steps and the resultant graph is the required SNI graph which contains G as an induced subgraph. ■

For example, an SNI graph constructed as in the proof of the above theorem containing K_4 as an induced subgraph is shown in Figure 8.

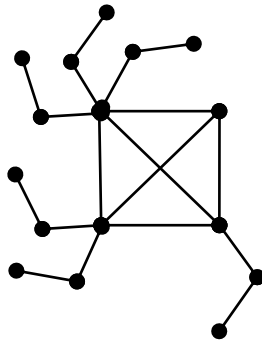


Figure 8

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