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Role of Hilbert Space in Sampled Data to Reduced Error Accumulation by Over Sampling Then the Computational and Storage Cost Increase Using Signal Processing On 2-Sphere Dimension"

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Abstract
Hilbert Space has wide usefulness in signal processing research. It is pitched at a graduate student level, but relies only on undergraduate background material. The needs and concerns of the researchers In engineering differ from those of the pure science. It is difficult to put the finger on what distinguishes the engineering approach that we have taken. In the end, if a potential use emerges from any result, however abstract, then an engineer would tend to attach greater value to that result. This may serve to distinguish the emphasis given by a mathematician who may be interested in the proof of a fundamental concept that links deeply with other areas of mathematics or is a part of a long-standing human intellectual endeavor not that engineering, in comparison, concerns less intellectual pursuits. The theory of Hilbert spaces was initiated by David Hilbert (1862-1943), in the early of twentieth century in the context of the study of "Integral equations". Integral equations are a natural complement to differential equations and arise, for example, in the study of existence and uniqueness of function which are solution of partial differential equations such as wave equation. Convolution and Fourier transform equation also belongs to this class. Geometric intuition plays an important role in many aspects of Hilbert space theory. Exact analogs of the Pythagorean theorem and parallelogram law hold in Hilbert space. At a deeper level, perpendicular projection onto a subspace that is the analog of "dropping the altitude" of a triangle plays a significant role in optimization problem and other aspects of the theory. An element of Hilbert space can be uniquely specified by its co-ordinates with respect to a set of coordinate axes that is an orthonormal basis, in analogy with Cartesian coordinates in the plane. When that set of axes is countably infinite, this means that the Hilbert space can also usefully be thought in terms of infinite sequences that are square summable. Linear operators on Hilbert space are ply transformations that stretch the space by different factors in mutually perpendicular directions in a sense that is made precise by the study of their spectral theory. In brief Hilbert spaces are the means by which the ordinary experience of Euclidean concepts can be extended meaningfully into idealized constructions of more complex abstract mathematics. However, in brief, the usual application demand for Hilbert spaces are integral and differential equations, generalized functions and partial differential equations, quantum mechanics, orthogonal polynomials and functions, optimization and approximation theory. In signal processing which is the main objective of the present thesis and engineering. Wavelets and optimization problem that has been dealt in the present thesis, optimal control, filtering and equalization, signal processing on 2- sphere, Shannon information theory, communication theory, linear and non-linear theory and many more is application domain of the Hilbert space.

Keywords: Hilbert Spaces, Differential Equations, Fourier Transform Orthogonal Polynomials, Optimization and Approximation Theory, Signal Processing On 2- Sphere

1.1 Introduction

Since its introduction in communication engineering by Shannon, the sampling theorem has played an important role in both mathematics and electrical engineering. The sampling theorem has derived for functions with valued in separable Hilbert space. The proof uses the concept of frames and frame operators in Hilbert space. One of the advantages of this theorem is that it allows us to derive sampling theorem associated with bounding values problems and some homogeneous integral equations, which in turn gives us a generalization of another sampling theorem.

The theorem says that if a signal (function) of time, t, is limited in its band width to W cycles/ second, it is completely determined by its values at a series of discrete point spaced 1/2W seconds apart and can be recovered at any time t by using its values at discrete set of points.

Mathematically this can be represented as

If f(t) is a function band limited to [-2□W, 2□W] that is

f(t) = (1/□2□) □-+□□F(□) eitW d□ (1.1)

for some F □ L2[-□,□], where □ = 2□□, then it can be reconstructed from its samples at points tk = k□/□, k = 0, ±1, ±2, ... via the formula

f(t) = □k□=-□ f(tk) {sin□(t-tk)/□(t-tk)} , t □ , (1.2)

Where the series converges absolutely and uniformly on compact set of the real line. This theorem has been generalized in many different ways.

[1] In one direction, the equally spaced points {tk}k□Z. Where Z is the set of integers, are replaced by non-uniformaly spaced points leading to the following generalization by Paley and Wiener. Let {tk}k□Z be sequence of real numbers such that

s ku□pZ |tk – (k□/□)| < □/4□ (1.3)

And let P(t) be the entire function defined by

P (t) = (t – t0) k□=1 (t – (t/tk)) (1 – (t/t-k)) (1.4)

Then for any f in the form equation (6.1) we have

f (t) = □k□=-□ f(tk) { P(t) / (t-tk) P'tk } (1.5)

with the series being absolutely and uniformly convergent on compact sets. If tk = (k□/□), then P(t) reduces to Sin□t/□ and equation (1.5) reduces to equation (1.2).

[2] In another direction, the Kernel function eit□ in the equation (1.1) is replaced by a more general Kernel k(□,t) leading the following generalization by Kramer. Let K(x,t) be the continuous function in t, such that as a function of x, k(x,t) □ L2(I) for every real number t, where I = [a,b], -□< a < b < □.

Assume that there exist a sequence of real numbers {tk}k□Z such that {K(x,tk)}k□Z is a complete orthogonal family of function in L2(I). Then for any f of the form

f(t) = □ab K(x,t) F(x) dx (1.6)

Where F □ L2(I), we have

f(t) = □k□=-□ f(tk) Sk*(t) (1.7)

Where

Sk*(t) = { □ab K(x,t) k—(—x—,—t—k) dx } / { □ab |K(x,tk)|2 dx } (1.8)

If I = [-□,□], tk = k□/□ and K(x,t) = eixt, then obviously yields

Sk*(t) = sin□(t-tk) / □(t-tk)

and hence equation (1.7) reduces to equation (1.2).

[3] One way to generate kernel K(x,t) and sampling points {tk}k□Z is to be consider the regular sturm-Lioville bounding value problem.

-y" + q(x) y = ty , x □ I = [a,b] (1.9)

y(a) cos□ + y'(a) sin□ = 0 (1.10)

y(b) cos□ + y'(b) sin□ = 0 (1.11)

where q is continuous on I. Then take K(x,t) to be the solution of the differential equation (1.9) and the initial condition (1.10) or the solution of (1.9) and (1.10) and take the sampling points {tk}k□Z to the eigen values of the problem since the eigen functions {K(x,tk)}t□Z form a complete orthogonal family in L2(I).

Although it is theoretically feasible to extend this procedure to more general self-adjoint boundary value problem associated with the nth order differential operators in practice this does not always work since existence of one single function K(x,t) which generates all the eigen functions of the problem when parameter t is replaced by the eigen values, is not always guaranted. For example

-y" = ty , x □ [0,□]

y(0) = y(□) aand y'(0) = y'(□)

are not generated by one single real-valued function. One possibility to circumvent this problem is to use the Green's function method described in previous chapter.

For many self-adjoint boundary value problems, the green functions can be written as

G(x,y,□) = □n□=1 {□n(x) □n(y) / (□-□n)} (1.12)

where {□n}n□=1 are the eigen value and {□n}n□=1 are the corresponding eigen functions. Thus Green's function method can also be used to derive sampling theorem associated with homogeneous.

The aim of the present thesis is to generalize some of the above results obtained to derive a sampling theorem for vector-valued functions. These functions take values in a separable Hilbert space H. One of the interesting ramifications of this generalisation is that allows us to obtain sampling theorem associated with boundary value problems and integral equations without restricting the sampling point to be the eigen values of the corresponding problem. In fact, the sampling poins will be arbitrary, except for a restriction on their growth rate. The idea of using Hilbert space concept in sampling theory used the concept of frames in a Hilbert space to derive sampling theorems for band-limited functions. In this chapter we also derive our sampling theorem using the concept of frames.

1.2 Preliminaries

Throughout the thesis the sets of complex and real numbers have been denoted by C and respectively and we have denoted as a separable Hilbert space with inner product <□> and norm ||f|| = □<f,f> for all f □ , here we have used ||x|| = □<x,x> for all x □ . The Fourier transform of a function f(t) is defined as

f^□(□) = (1/□2□) □-+□□ f(t) eitw dt (1.13)

so that the inverse transform is given by

f(t) = (1/□2□) □-+□□ f^□(□) e-itw d□ (1.14)

Provided that the integral exist.

Let B□2 denote the class of all entire function f of exponential type at most □ □ L2( ) when restricted to the real axis that is f □ B□2 if and only if

|f(z)| □ s xu□p□ |f(x)| exp (□|y|) (1.15)

where Z = x + iy and

□-+□□ |f(x)|2 dx < □ (1.16)

The well-known Paley-Wiener theorem states that f □ B□2 (□>0) is and only if

f(t) = (1/□2□) □-+□□ F(w) eitw dw (1.17)

for some F □ L2[-□,□]. The class B□2 is usually called the Paley-Wiener class of entire functions.

Let G = {gn} be a sequence in Hilbert space . We say that G is a frame if there exist two number A, B > 0 such that for every f □ , then

A||f||2 □ □n |<f,gn>|2 □ B ||f||2 (1.18)

The two numbers A and B are called frame bounds. The frame is said to be light if A = B and is exact if it ceases to be a frame whenever any single element is detected from the frame. Frames are complete if <f, gn>

= 0, for all n, then ||f|| = 0 and consequently f = 0, we have already derived in previous section, G is said to be bounded if there exist two non-negative numbers C and D such that C □ ||gn|| □ D for all n. It is known that a frame is exact if and only if it is a bounded and unconditional basis. A basis G is said to be unconditional if

□n Cn gn □ □ □ |Cn| gn □ (1.19)

with every G, we associate a frame operator S defined by

Sf = □n < f, gn > gn (1.20)

where S is a bounded linear operator with AI □ S □ BI, and that S is invertible, where AI □ S □ BI means A < x, x > □ < Sx, x > □ B (x,x) for all x □ .

The inverse frame operator S-1 has the following properties

[1] B-1I □ S-1 □ A-1I, and

[2] {S-1gn} is also frame with frame bounds B-1 and A-1.

1.3 The main result:

Let {□n}n□=1 be a sequence of complex numbers, non of which is with the point of at infinity as its only limit point. The convergence exponent □ of the sequence {□n}n□=1 is defined as

□ = inf { □ □ : □n□=1 (1/|□n|□) < □ } (1.20)

Let us further assume that {□n}n□=1 has finite convergence exponent that is 0 □ □ < □ and let p denote the smallest positive integer such that □n□=1 (1/|□n|p+1) converges.

Also let,

□n□=1 (1 – (□/□n)) exp [(□/□n) + (1/2) (□/□n)2 + ...

+ (1/p) (□/□n)p] if p = 1, 2, ...

P(□) = (1.21)

□n□=1 (1 – (□/□n)) if p = 0.

We can include zero as one of the terms of the sequence {□n) and in this case we will denote it by □0 and redefine P(□) as

□ □n□=1 (1 – (□/□n)) exp [(□/□n) + (1/2) (□/□n)2 + ...

+ (1/p) (□/□n)p] if p = 1, 2, ...

P(□) = (1.22)

□ □n□=1 (1 – (□/□n)) if p = 0.

It can be shown that P(□) is an entire function in □ of order equal to □. Let {gn}n□=1 be a frame in a separable Hilbert space H and S be its frame operator. Then the dual frame {S-1gn}n□=1 is denoted by {g*}n□=1. If {gn}n□=1 is exact, {gn}n□=1 and {gn* }n□=1 are biorthonormal, that is

0 if m □ n

< gm, gn* > = □m,n = (1.23)

1 if m = n

For each fixed □ □ □1, □2, ... then we define operator as

L□ = P(□) □n□=1 { < □, gn > / (□ - □n) } gn* (1.24)

Then the above operator applied on Hilbert space H in the usual way we have

L□f = P(□) □n□=1 { < f, gn > / (□ - □n) } gn* , f □ H (1.25)

and for □ = □k, k = 1, 2, ... then

L□k = P'(□k) < □, gk > gk* (1.26)

Now for fixed f □ , define F:C □ H by

L□f = P'(□) < f, g > gn* = F(□)

therefore F(□) = L□f (1.27)

Having basic ideas we are in a position to prove sampling theorem.

Statement [1] for each fixed □ □ C, L□ is a bounded linear operator on , and if k is a compact subset of the complex □-plane, then the set of operation {L□}□□k is uniformly bounded.

[2] F□ is continuous vector valued function that is completely determined by its values {F(□n)}n□=1 and can be reconstructed from these values according to the formula.

F(□) = □n□=1 { P(□) / (□ - □n) P'(□n) } F(□n)

Proof: The linearity of L□ is trivial once we show that L□ is well defined. We prove first statement. But first we have to recall that for any □□ , ||□|| can be given as

||□|| = s||uhp||=1 | < □, h > | (1.28)

Let Sm,□f = P(□) □km=1 { < f, gk > / (□ - □k) } gk* (1.29)

Then for 1 □ m □ n and the Cauchy-Schwarz inequality as explained in previous chapter. We have

||Sm,□f – Sn,□f||2 = s||uhp||=1 | < Sm,□f – Sn,□f > |2 (1.30)

= s||uhp||=1 | P(□) □kn=m+1 {< f, gk > / (□ - □k) } < gk* , h > |2

□ |P(□)|2 s||uhp||=1 [□kn=m+1 { |< f, gk >|2 / |□ - □k|2 }]

[□kn=m+1 |< gk* , h > |2] (1.31)

Since {gk* }k□=1 is a frame with frame bonds B-1 and A-1, yield

||Sm,□f – Sn,□f||2 □ |P(□)|2 [□kn=m+1 { |< f, gk >|2 / |□ - □k|2 }] A-1

(1.32)

Let K be a compact subset of complex □-plane and □— = {□i1, ..., □iq} be the set of □ i' s that lies inside k. Also let sequence {□n}n□=1 is denoted by □ and the distance between K and □ - □— by □. Then for any □□ K and □k □ □ - □—, then we have

s □u□pk |P(□)/(□-□k)| □ (1/□) s □u□pk |P(□)| = (1/□) ||P(□)||k, (1.33)

where ||P||k = s □u□pk |P(□)|.

hi = P(□) / (□-□i) , where i = i1, i2, ..., iq

Clearly, hi is an analytic function except possibly at □ = □i, but since P has a zero at □ = □i, hi is actually an entire function.

Therefore,

m □a□xk |hi(□)| = ||hi||k is finite.

Let C = max {||hi1||k, ..., ||hiq||k} and

C(K) = max {C, ||hi||k / □} (1.34)

Now combining equation (1.34) and 1.32) yields

||Sm,□f – Sn,□f||2 < C2(K) [□kn=m+1 |< f, gk >|2] A-1 □ 0 (1.35)

as m,n □ □.

Thus {Sm,□f}m□=1 is a Cauchy sequence and hence

lmi□m□ Sm,□f = L□f (1.36)

Now letting n □ □ in equation (1.35), it follows that Sm,□ converges to L□f uniformly on compact subsets of the complex □-plane.

On repeating the above argument once more leads to

||F(□)||2 – ||L□f||2 = s||uhp||=1 | < L□f, h > |2

= |P(□)|2 s||uhp||=1 [□k□=1 {|<f, gk>|2 / |□-□k|2}] [□k□=1 |<gk,h>] (1.37)

□ A-1 C2 (k) [□k□=1 | <f, gk> |2]

□ A-1 B C2 (k) || f ||2 (1.38)

Which shows that for fixed □, L□ is continuous linear operator on Hilbert space . In fact equation (1.38) also shows that the family of operator {L□}□=k is uniformly bounded. Thus part one has been proved.

Now in order to prove part [2]. From [1], it follows that F(□) is well defined. Now we have show that it is continuous. Since P(□) is an entire function, it suffices to show that

G(□) = (1/P(□)) F(□) is continuous.

Let □* □ C - □ and denote the distance between □* and □ by 2□. Let D□(□*) = { □: |□-□*| □ □ } be the close distance with centre □* and radius □. For any □* □ D□(□*), yields

||G(□) - G(□*)||2 = s||uhp||=1 | <G(□) - G(□*),h> |2 (1.39)

□ s||uhp||=1 [□k□=1 |□*-□| / |□-□k| |□*-□k|] |< f, gk >|2

[□k□=1 |< gk*, h >|2] (1.40)

□ A-1 |□*-□|2 □k□=1 [1 / |□-□k| |□*-□k|]2 |< f, gk >|2

□ A-1 □-4 B ||f||2 |□*-□|2 (1.41)

Thus,

||G(□) - G(□*)||2 □ A-1/2 □-2 B1/2 ||f|| |□*-□|

□ (B/A)1/2 (||f||/□2) |□*-□| □ 0 (1.42)

as □ □ □*

Since F(□) is continuous for each □ □ C – □ and

l□i□m□n F(□) = F(□n) = P'(□n) < f, gn > gk* (1.43)

So, F is continuous everywhere. Proved.

and Finally statement [2] follows from equation (1.25) and equation (1.43), that is

F(□) = □n□=1 [P(□) / (□-□n) P'(□n)] F(□n) (1.44)

Statement [2] be also proved as follows.

Let the eigen vector of L form an orthonormal basis, hence they form an exact light frame with frame bounds = 1 and gn*s = gn. As a specified case, let = L2(I), where I = [a,b], -□ < a < b < □, and

(Lf) (x) = □ab k(x,□) f(□) d□, f □ L2(I) (1.45)

If K is real, symmetric and in L2(Q), where Q = I x I, then L is a self-adjoint compact operator on L2(I). In addition, if the equation

□ab k(x,□) f(□) d□ = 0 (1.46)

has only the trivial solution f = 0, then the eigen functions {gn}n□=1 of L form an orthonormal basis for L2(I) and we have the following sampling theorem. For any sequence numbers {□n}n□=1 satisfying the assumption of the theorem [1] if

F(□) = □ab f(x) R(x, □, □) dx (1.47)

where R(x, □, □) = P(□) □n□=1 gn(x) gn(□) / (□-□n)

as □ is a fixed point in I, then

F(□) = □n□=1 F(□n) { P(□) / (□-□n)P'(□n) } (1.48)

when the □n's are taken as eigen values of L, then R/P becomes the resolvent associated with the integral equation and in case the self adjoint boundary value problem it becomes the Green function of the problem.

1.4 Inversion formula

In this section we have derived an inversion formula for the vector valued function F(□) by using Bochner integral of F. For this purpose, we need to restrict the growth rate of the sequence {□n}n□=1 and require it to have ±□ as its only limit points. Therefore, it is more appropriate to use integers as the index set for the sequence instead of natural numbers. For this let us assume that there exists a sequence of functions {hn(x)}n□Z such that

P(□) / (□-□n)P'(□n) = □-+□□ hn(x) ei□x dx (1.49)

0 if n □ m

And □-+□□ hn(x) ei□mx dx = (1.50)

1 if n = m

Set

Bn(□) = □-+□□ ei□nx e-i□x dx (1.51)

Hence, ei□nx X[-□,□] (x) = (1/2□) □-+□□ Bn(□) ei□x d□ (1.52)

Where X[-□,□] is the characteristic function [-□,□] and the integral converges in the sense of L2,

Let KN (□) = (1/2□) □KN=-N BK(□) / P'(□K) (1.53)

From equation (1.49) – (1.51) and parseval inequality, we have

(1/2□) □-+□□ {P(□) BK(□) / (□-□n) P'(□n)} d□

= □-+□□ hn(x) ei□kx dx = □n,k (1.54)

But we know that

F(□) = □n□=1 { P(□) / (□-□n) P'(□n) } F(□n) (1.55)

Thus from equation (1.54) and (1.55) we obtain

□-+□□ F(□) KN(□) d□ = □n□=-□ □□KN=-N (F(□n) / P'(□k)) [(1/2□)

□-+□□ {P(□) BK(□) / (□-□n) P'(□n)} d□] (1.56)

= □nN=-N (F(□n) / P'(□k))

= □□nN=-N < f, gn > gn* (1.57)

Here the last inequality follows from the expression as

l□i□m□ F(□) = F(□n) = P'(□n) < f, gn > gn*

Now interchanging the summation and integration sign is possibly by the dominated convergence theorem for Bochner integrals; since

□-+□□ ||F(□)||2 d□ □ A-1 □n□=-□ |< f, gn >|2 □-+□□ |P(□)/(□-□n)|2d□

□ A-1 BD ||f||2 < □ (1.58)

Now taking limits in equation ((1.57) as N □ □, yields

f = l□i□m□ □-+□□ F□ KN(□) d□ (1.59)

If the series (1/2□) □K□=-□ (BK(□) / P'(□K)) converges to a square integrable function K□, then equation (1.59) becomes

f = □-+□□ F(□) K(□) d□

1.5 Sampling on unit sphere or 2-sphere S2

Let us consider that we wish to numerically compute the spherical harmonic coefficient of a function f □ L2(S2), (f) lm , and we have "sufficient" samples of f at points (□j, □k) over the entire 2-sphere, then (f) lm □ d□ d□ □j □K f(□j, □k) Y—lm—(—□—j—□—k) Sin□j (1.60)

where d□, d□ are the sampling resolution. This principle is fine, but there may be some computational issue with it. [First, we cannot sure about the numerical accuracy and the stability of this technique for a given number of samples. This is especially relevant if the results are going to fed to another sophisticated algorithms and the error accumulation may occur.

Second, if we attempt to improve accuracy by over sampling, then the computational and storage costs increase.

So, it is desirable to have more efficient and provably accurate techniques for signal processing and relevant computations on 2-sphere. Therefore, this has been an active and practically relevant area of research. Let specially, given signal f is uniformly sampled with N points along the colatitude □ and longitude □, a widely used method has been proposed in that turns the above approximation into an exact computation for all degrees smaller than N/2, where N is the number of samples along the co-latitude and longitude. This is accomplished using a "quadrature" rule with approximately chosen weights. Here we briefly reviewed the results, as they can be useful in applications involving signal processing on 2-sphere].

1.5.1 Sampling distribution

Sampling on unit sphere or 2-sphere can be best understood using sampling distribution as well as sampling theorem. For example let us assume that we have a signal f □ L2(S2) such that (f) lm = 0 for all l > Lf. That is, there are a maximum of (Lf+1)2 non-zero spherical harmonic coefficients for f. Now any (Lf+1)2 independent samples as our sampling distribution (that is, (Lf+1)2 independent co-latitude and longitude pairs) lead to a linear system in the unknown spherical harmonic co-efficients, which can be solved exactly via Matrix inversion. The real issue have, though, is having a sampling distribution that leads to a numerically well-conditional system, match real world sampling schemes and more importantly leads to a reasonable computational complexity.

We can uniformly sample both the co-latitude and longitude angles with N samples, satisfying N □ 2(Lf+1) at point, that is

□j = □j/N , j □ {0, 1, 2, ..., N-1}

and (1.61)

□k = 2□k/N , k □ {0, 1, 2, ..., N-1}

obviously, there are more samples around the poles than the equator and this needs to be compensated for with some proper weighting. The specific weighted sampling distribution function have been proposed and corrected for minor scaling error, as

S(□,□) = (2□□2/N) □jN=-01 □kN=-01 □j(N) □(□-□j) □(□-□k) (1.62)

where the coefficients □j(N) must satisfy the following sets of N equations

□jN=-01 □j(N) Pl(cos□j) = □l,0 , l □ {0, 1, 2, ..., N-1} (1.63)

where Pl(□) is the Legendre polynomial of degree l. If N is choosen to be a power of 2, then the coefficients have a closed form solution as

□j(N) = (2□□2/N) Sin(□j/N) □kN=/02 -1 (1/2k+1) Sin[(2k+1)(□k/N)]

j □ {0, 1, 2, ..., N-1} (1.64)

1.5.2 Sampling theorem on 2-sphere

The sampling theorem on 2-sphere has already defined as, if a function f(□,□) be band limited on 2-sphere such that (f)lm = 0 for all degrees l □ {Lf+1, Lf+2, Lf+3 ...} and let the signal be sampled at least N = 2(Lf+1) uniform points on the sphere in both co-latitude and longitude according to the equation (1.61) and signal values be denoted by f(□j,□k). The spherical harmonic coefficients of signal f up to degree l can be exactly revolved using.

(f)lm = (2□□2/N) □jN=-01 □kN=-01 □j(N) f(□j,□k) Y—lm—(—□—j—□—k)
(1.65)

for degree l □ {0, 1, ..., Lf} and order m □ {-l, -l+1, ... l} where □j(N) satisfy the equation (6.63). Any possible aliasing will occur for degrees beyond Lf.

1.6 Error Calculation

Error calculations in terms of reliability and computational cost. Let us consider the samples of a signal f at points □j = □j/N and □k = K (2□/N) for j, k = {0, 1, 2... N-1} and let N = 4 then f(□j,□k) is given by

	0.5481	0.4606	0.9703	0.4558
	0.7807	0.9209	0.7523	0.5205
f(□j,□k) =				
	0.5589	0.8862	0.0155	0.0889
	0.3017	0.8795	0.9923	0.7851

Where horizontal direction represents change in co-latitude □ and vertical direction corresponds to change in longitude □.

Now using exact method or sampling theorem equation (6.65) on 2-sphere, that is

(f)lm = (2□□2/N) □jN=-01 □kN=-01 □j(N) f(□j,□k) Y—lm—(—□—j—□—k)

where Y—lm—(—□—j—□—k) = (-1)m Yl-m(□j,□k)

and Ylm(□j,□k) □{(2l+1(l-m)!) / 4□(l+m)!} Plm(Cos□j) eim□k

where Plm(x=Cos□) is Associated Legendre Polynomial for low degree l □ {0, 1, 2, 3, 4} and order m □ {-l, -l+1, ..., -l}. The vector of spherical coefficients have been found as f = (2.2829, 0.3304+i0.1439, 0.4692, -0.3304+i0.1439)', where ' represents transpose of vector matrix. Similarly, when we have used equation (6.60), that is

(f)lm □ d□□d□ □j □k f(□j,□k) Y—lm—(—□—j—□—k) Sin□j

the vector of spherical coefficients have been found as

f = (2.1809, 0.3945+i0.1499, 0.3909, -0.3945+i0.1499)'

when we compared the difference between exact method and approximate method, the difference is non-negligible. The normalized err, which is defined as

□^ □l□=0 □ml=-l | (f)lm – (f^)lm |2) / (□l□=0 □ml=-l | (f)lm |2)1/2

which have found □^ = 0.0662 and % □^ = 6.62

Now when we use f as our reference to oversample the signal with N = 8 as

f2(□j,□k) = □l□=0 □ml=-l | (f)lm Y(□j,□k)

Then we get the some spherical harmonic coefficient as in f, if we feed f2 in equation (6.65) with normalized error.

□^ = 2.332 x 10-16.

We can also improve the accuracy of direct calculation f = (2.2535, 0.3305 + i0.1440, 0.4508, -0.3305 + i0.1440)'. Here normalized error has been reduced to □^ = 0.0145. Thus in brief, we can say the exact method is superior to the direct computation method in terms of reliability and computational cost.

In evidence of the above statement let us compare the exact method (1.65) and the approximate method (1.62) for a reference signal, whose spherical harmonic coefficient is known. We take f(□,□) = Y2-1 (□,□) and hence (f)lm = □2,l □-1,m and also take N = 8 to evaluate Y(□j,□k). Now using equation (6.64) as given below

□j(N) = (2□□2/N) Sin(□j/N) □kN=/02 -1 (1/2k+1) Sin[(2k+1)(□k/N)] j □ {0, 1, 2, ..., N-1}

□ has been calculated for N = 8 as

□ = (0, 0.1258, 0.1751, 0.2782, 0.2559, 0.2782, 0.1751, 0.1258)'

In simulation, the error is defined as

□ (□l□=0 □ml=-l | (f)lm – □2,l □-1,m |2)1/2

and from simulation the error has been calculated to be 2.1314 x 10-16 using exact method, whereas the error using approximate method with some number of sample N = 8 was found to be 1.7000 x 10-3.

Now when we increase the number of sample N'=32 which reduces the error to 5.8463 x 10-6.

2.1 Conclusion

Since sampling in a Hilbert space introduction in communication engineering by Shannon in 1948, the Whittaker-Shannon-Kolel'nikov sampling theorem has played an important role in both mathematics and electrical engineering. Hence an analog of the Wittaker-Shannon Kolel'nikov sampling theorem has been derived for functions with the values in a separable Hilbert space. One of the most important consequences of the derived sampling theorem is that it allows us to derive sampling theorems associated with boundary value problems and some homogeneous integral equations, which in turn gives us a generalization of another sampling theorem.

As the aim of this thesis is to generalize the results obtained to derive a sampling theorem for vector valued functions. These functions take values in a separable Hilbert space . One of the interesting ramifications of this generalization is that it allows us to obtaining sampling theorems associated with boundary value problems and integral equations without restricting the sampling points to be the eigen values of the corresponding problem. In fact, the sampling points will be arbitrary, except for a restriction on their growth rate.

Some problems in sampling on 2-sphere has been considered to numerically compute the spherical harmonic coefficients of a given functions and we have taken "sufficient" samples of f at points □j and □k over the entire 2-sphere and have been derived an approximate equation and compared it with sampling theorem on 2-sphere.

In article, we have compared the error results using approximate method and exact method. We came into conclusions that, the exact method is superior to direct method in terms of reliability and computation cost.

For example for a reference signal, whose speherical harmonic coefficient is known & considered sample points as 8 and calculated error which has been found 2.1314 x 10-16 in simulation on exact method, but it has been computed on approximate method the error has been found 1.7 x 10-3.

When the number of samples has increased from 8 to 32. Then the error has been reduced to 5.8463 x 10-6. Thus number of sample also plays an important role to reduce the error. In second case, the normalized error has been found equal to 0.0662 when we have taken N = 4. When N has been increased 8 and compared error using exact and approximate method it has been concluded that normalized error has been reduced to 0.0145.

In summary, the exact method is superior to direct method of computation in terms of

[a] Reliability performance and

[b] Computation cost

3.1 Future Scope

In order to find optimally concentrated signals in spatial or spectral domains has many applications in practice, which are similar in spirit and can be as diverse as applications of the concentration problem in time-frequency domain. For example, we may need to smooth a signal on the sphere to reduce the effects of noise or high spectral components by performing local convolutional averaging or filtering, which will be the topic of future scope for further research work as optimally concentrated window's that allow accurate joint spatio-spectral analysis of signals on the sphere.

4.1 References

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