International Journal of Scientific Research and Management (IJSRM)
||Volume||08||Issue||12||Pages||M-2020-290-295||2020||
Website: www.ijsrm.in ISSN (e): 2321-3418
DOI: 10.18535/ijsrm/v8i12.m01

Stochastic and Deterministic Model of the Spread of a Disease in Age-Structured Population

Yéo Yardjouma

Abstract
We model the spread of an epidemic which depends on age and time. The behavior of the model obtained was the object of our studies. This studies shown that on each characteristic curve, the central limit theorem and the law of large numbers were satisfied. In this paper, we write a stochastic model to studies a disease which is begin in a population which is not large. we study overall the stochastic model obtained by varyingthe age and time together. We prove that the law of large numbers is satisfied.

Introduction

The models of the infectious diseases started to be set up a long time ago. Thus, several mathematicians[2][6] [7] [3] model the spread of an epidemic disease. Their idea consists to subdivid the population in three compartments. Compartment S denote the compartment of susceptible, I the compartment of infectious and R the compartment of removed (i.e. healed and immune). During the spread of an epidemic, the rate contamination and that of cure can depend on the age of individuals. It is the case of Covid-19 where old people are most fragile. That will enable us to build the compartmental model in age-structured. In this context, we decided to contribute our share by making recourse to mathematical tools to predict the evolution of an epidemic which depends on age and time. This tool of decision can help the authorities to plan a certain number of actions in order to control the spread of an epidemic. The model obtained is deterministic type and present a limit when the epidemic starts in a group of few individuals. That will enable us to build his stochastic version. Then, our work is to see whether stochastic model obtained present better properties. In other words it is a question of seeing whether the stochastic model converges towards the deterministic model when the size of the population becomes increasingly large.

The Models Deterministic model The spread of a disease in a population is a dynamic phenomenon where the number of individuals Sains, infected, and Cured (or died) can move according to time, age and contacts between people Saines and infected. This phenomenon can be modelled by mathematical models. These models are an approximation of reality, they present certainly uncertainties but have watch their effectiveness in the prediction of the epidemics. These models are generally used as tools of decision-making. In this work, we will use this tool to build models in age. One of the model used in the scientic literature to model the propagation of a epidemic is SIR model. It was introduced by Kermack and Mckendrick in 1927 after a study on the spread of the Spanish in influenza in 1918, which made between 50 and 100 million deaths. we denote by S 1 for the healthy individuals : the people likely to be infected, I for the infected individuals and R for the infected individuals who cure and develops a permanent immunity or die. We find a SIR model which depends on age and time.

$$\begin{cases} \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) S(a,t) = -\lambda(a,t)S(a,t) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) I(a,t) = \lambda(a,t)S(a,t) - \gamma(a,t)I(a,t) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) R(a,t) = \gamma(a,t)I(a,t) \end{cases} \quad (1)$$

) let us aM the maximum experiment of life, S(a, t) the number of healthy individuals who at the time t has age a, I(a, t) the number of infected individuals who at the time t has age a and R(a, t) the number of infected individuals who cure and develops a permanent immunity or die who at the time t has age a. We consider the following system (2) where s, i and r represent the proportion of susceptibles respectively, infected and removed

$$\begin{cases} \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) s(a,t) = -\lambda(a,t)s(a,t) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) i(a,t) = \lambda(a,t)s(a,t) - \gamma(a,t)i(a,t) \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right) i(a,t) = \gamma(a,t)i(a,t) \end{cases} \quad (2)$$

The state space is give by

$$\Omega_0 = \left\{(s,i,r) \in L^1_+(0,a_M) \times L^1_+(0,a_M) \times L^1_+(0,a_M) : s+i+r = 1\right\}$$

Where S(a, t) = NsN (a, t), i(a, t) = N iN (a, t), R(a, t) = NrN (a, t), N is the population size. Our objective is to show that sN (a, t) converge towards s(a, t), iN (a, t) converge towards i(a, t), rN (a, t) converge towards r(a, t) when N goes to $+\infty$. Theorem 2.1 The dynamics of healthy individuals, infected individuals and infected individuals who cure and develops a permanent immunity or die satisfied the following

$$\begin{cases} s(a,t) = s(a-t,0)e^{-\int \lambda(a,t)dt} \\ r(a,t) = r(a-t,0) + \int_0^t \gamma(a-t+u),u)i(a-t+u),u)du \\ i(a,t)) = 1 - s(a,t) - r(a,t) \end{cases} \quad (3)$$

Proof

Let us consider equation

$$\frac{\partial s}{\partial t}(a,t) + \frac{\partial s}{\partial a}(a,t) = -\lambda(a,t)s(a,t) \quad (4)$$

$\frac{dt}{du} = 1 \Rightarrow dt = du \Rightarrow \int dt = \int du + c_1 \Rightarrow t = u + t_0.$
$\frac{da}{du} = 1 \Rightarrow da = du \Rightarrow \int da = \int du + c_2 \Rightarrow a = u + a_0.\ c_1, c_2 \in \mathbb{R}.$

If we consider the function s which solve the equation (4) us a function of u (ie) s(u) = s(a(u), t(u))

$$\frac{ds}{du} = \frac{\partial s}{\partial a}\frac{da}{du} + \frac{\partial s}{\partial t}\frac{dt}{du} = \frac{\partial s}{\partial a} + \frac{\partial s}{\partial t} = -\lambda(a(u),t(u))s$$

$$\frac{ds}{du} = -\lambda(a(u), t(u))s \Longleftrightarrow \frac{ds}{s} = -\lambda(a(u), t(u))du$$

$$\int \frac{ds}{s} = -\int \lambda(a(u), t(u))du + c_3 \Rightarrow \ln(s) = -\int \lambda(a(u), t(u))du + c_3$$

If we x s(0) = s0, then we nd s = s0e $^{-\int_R \lambda(a(u),t(u))du}$.

Thus the caracteristics curves are

$$u \longmapsto (a(u), t(u), s(u)) = \left(u + a_0, u + t_0, s_0 e^{-\int \lambda(a(u),t(u))du}\right)$$

If we consider the initials conditions t0 = 0 and f(a0) = s0, where f ∈ C1 (R, R+) and f(a − t) = s(a − t, 0), then the caracteristics curves are

$$u \longmapsto (a(u, a_0), t(u, a_0), s(u, a_0)) = \left(u + a_0, u, f(a_0)e^{-\int \lambda(a(u,a_0),u)du}\right)$$

It is possible to find the invers function of

$$u \longmapsto (a(u, a_0), t(u, a_0))$$

In fact,

$$\begin{cases} u + a_0 = a \\ u = t \end{cases} \Rightarrow \begin{cases} a_0 = a - t \\ u = t \end{cases} \tag{5}$$

We obtain the solution s(a, t) = f(a − t)e − R λ(a,t)dt of the equation (4)

We consider the equation

$$\frac{\partial r}{\partial t}(a, t) + \frac{\partial r}{\partial a}(a, t) = \gamma(a, t)i(a, t) \tag{6}$$

By changing the variable $v(t) = r(g_0(t), t)$, where $g_0(t) = a_0 + t = a$

$$\frac{dv}{dt}(t) = \frac{dg_0}{dt}(t)\frac{\partial r}{\partial g_0}(g_0(t), t) + \frac{dt}{dt}(t)\frac{\partial r}{\partial t}(g_0(t), t) = \frac{\partial r}{\partial t}(a, t) + \frac{\partial r}{\partial a}(a, t) = \gamma(a, t)i(a, t)$$

$$\frac{dv}{dt}(t) = \gamma(a, t)i(a, t) = \gamma(g_0(t), t)i(g_0(t), t)$$

$$dv(t) = \gamma(g_0(t), t)i(g_0(t), t)dt \Rightarrow v(t) = v(0) + \int_0^t \gamma(g_0(u), u)i(g_0(u), u)du$$

$$v(t) = v(0) + \int_0^t \gamma(g_0(u), u)i(g_0(u), u)du = v(0) + \int_0^t \gamma(a_0 + u), u)i(a_0 + u), u)du$$

$$v(t) = v(0) + \int_0^t \gamma(a_0 + u), u)i(a_0 + u), u)du = r(a_0, 0) + \int_0^t \gamma(a - t + u), u)i(a - t + u), u)du$$

$$r(a, t) = r(a - t, 0) + \int_0^t \gamma(a - t + u), u)i(a - t + u), u)du$$

This last result is the solution of the equation (6). By using the stability set Ω0, s(a, t) + i(a, t) + r(a, t) = 1 we nd i(a, t)) = 1 − s(a, t) − r(a, t).

Stochastic Model

Let P1 and P2 two Poisson process. The stochastic model is given by the system

$$\begin{cases} s(a,t) = P_1\left(s(a-t,0)e^{-\int\lambda(a,t)dt}\right) \\ r(a,t) = P_2\left(\int_0^t \gamma(a-t+u,u)i(a-t+u,u)du\right) \end{cases} \tag{7}$$

Law of Large Numbers

Let us note $s_N(a,t) = \frac{s(a,t)}{N}$, $i_N(a,t) = \frac{i(a,t)}{N}$, $r_N(a,t) = \frac{r(a,t)}{N}$

$$\begin{cases} s_N(a,t) = P_1\left(s_N(a-t,0)e^{-\int\lambda(a,t)dt}\right) \\ r_N(a,t) = P_2\left(\int_0^t \gamma(a-t+u,u)i_N(a-t+u,u)du\right) \end{cases} \tag{8}$$

$M_1(t) = P_1(t) - t$, $M_2(t) = P_2(t) - t$. M_1 and M_2 are two martingales.

$$\begin{cases} s_N(a,t) = \frac{1}{N}M_1\left(Ns_N(a-t,0)e^{-\int\lambda(a,t)dt}\right) + s_N(a-t,0)e^{-\int\lambda(a,t)dt} \\ r_N(a,t) = \frac{1}{N}M_2\left(N\int_0^t \gamma(a-t+u,u)i_N(a-t+u,u)du\right) + \int_0^t \gamma(a-t+u,u)i_N(a-t+u,u)du \end{cases} \tag{9}$$

Consider the process $\mathcal{M}_1(a,t) = \frac{1}{N}M_1\left(Ns_N(a-t,0)e^{-\int\lambda(a,t)dt}\right)$ and
$\mathcal{M}_1(a,t) = \frac{1}{N}M_2\left(N\int_0^t \gamma(a-t+u,u)i_N(a-t+u,u)du\right)$.

Let

$$\mathcal{F}_t^a = \sigma\{(\gamma(a-t+u,u)i_N(a-t+u,u), \lambda(s_N(a-t,0)e^{-\int\lambda(a,t)} : 0 \le u \le t, a_M \ge a-t \ge 0\}.$$

Then we have :

Lemma 2.1 .
$\{\mathcal{M}_1(a,t), t \ge 0, a_M \ge a \ge 0\}$ and $\{\mathcal{M}_2(a,t), t \ge 0, a_M \ge a \ge 0\}$ are two $\mathcal{F}_t^a$-martingales which satisfie

$$\mathbb{E}\left(\mathcal{M}_1(a,t)\right) = 0 \text{ and } \mathbb{E}\left(\mathcal{M}_1(a,t)\right)^2 = \frac{1}{N}\mathbb{E}\left(s_N(a-t,0)e^{-\int\lambda(a,t)dt}\right)$$

$$\mathbb{E}\left(\mathcal{M}_2(a,t)\right) = 0 \text{ and } \mathbb{E}\left(\mathcal{M}_2(a,t)\right)^2 = \frac{1}{N}\mathbb{E}\left(\int_0^t \gamma(a-t+u,u)i_N(a-t+u,u)du)\right)$$

For the proof of this lemma, the reader can refer to the author[1].

Theorem 2.2 ***(Law of Larges Numbers)***
If $s_N(a-t,0) \to s(a-t,0)$ and $r_N(a-t,0) \to r(a-t,0))$ when $N \to +\infty$, then

$$\sup_{(a,t)\in[0;a_M]\times[0;\omega]}\{|s_N(a,t) - s(a,t)|\} \to 0$$

and

$$\sup_{(a,t)\in[0;a_M]\times[0;\omega]}\{|r_N(a,t) - r(a,t)|\} \to 0$$

when $N \to +\infty$

Where $\{(s(a,t), r(a,t))\}$ solve the system

$$\begin{cases} \frac{\partial s}{\partial t}(a,t) + \frac{\partial s}{\partial a}(a,t) = -\lambda(a,t)s(a,t) \\ \frac{\partial r}{\partial t}(a,t) + \frac{\partial r}{\partial a}(a,t) = \gamma(a,t)i(a,t) \\ s(a-t,0) = s_0, \quad r(a-t,0) = r_0 \end{cases} \tag{10}$$

proof

$s_N(a,t) = s_N(a_0,0) + \int_{a_0}^{t}\int_{0}^{t} s_N(u,v)dudv$ and $s(a,t) = s(a_0,0) + \int_{a_0}^{t}\int_{0}^{t} s(u,v)dudv$

$$\begin{aligned} s_N(a,t) - s(a,t) &= s_N(a_0,0) - s(a_0,0) + \int_{a_0}^{t}\int_{0}^{t} s_N(u,v)dudv - \int_{a_0}^{t}\int_{0}^{t} s(u,v)dudv \\ &= s_N(a_0,0) - s(a_0,0) + \int_{a_0}^{t}\int_{0}^{t} \left[s_N(u,v) - s(u,v)\right] dudv \end{aligned}$$

let as $\varphi_N \in \mathcal{C}(\mathbb{R}^2,\mathbb{R})$ where $\varphi_N(a,t) = s_N(a,t) - s(a,t)$. Then

$$\begin{aligned} \varphi_N(a,t) &= \varphi_N(a_0,0) + \int_{a_0}^{a}\int_{0}^{t} \varphi_N(u,v)dudv \\ &= \sum_{k=0}^{+\infty} \varphi_N(a_0,0)\frac{(t(a-a_0))^k}{k!^2} \end{aligned}$$

$$\begin{aligned} \varphi_N(a,t) &\leq \varphi_N(a_0,0)\sum_{k=0}^{+\infty}\frac{(t(a-a_0))^k}{k!} \\ &\leq \varphi_N(a_0,0)e^{t(a-a_0)} = \varphi_N(a-t,0)e^{t^2} \end{aligned}$$

$$\begin{aligned} |\varphi_N(a,t)| &\leq \left|\varphi_N(a-t,0)e^{t^2}\right| \\ &\leq \left|e^{\omega^2}\varphi_N(a-t,0)\right| \end{aligned}$$

If $s_N(a-t,0) \to s(a-t,0)$ when $N \to +\infty$, $\left|e^{\omega^2}\varphi_N(a-t,0)\right| \longrightarrow 0 \quad$ when $\quad N \longrightarrow +\infty$

By using the same reasoning we proof :

If $r_N(a-t,0) \to r(a-t,0))$ when $N \to +\infty$ let as $\psi_N \in \mathcal{C}(\mathbb{R}^2,\mathbb{R})$ Where $\psi_N(a,t) = r_N(a,t) - r(a,t)$.

$$\begin{aligned} |\psi_N(a,t)| &\leq \left|\psi_N(a-t,0)e^{t^2}\right| \\ &\leq e^{\omega^2}\psi_N(a-t,0) \longrightarrow 0 \quad \text{when} \quad N \longrightarrow +\infty \end{aligned}$$

Concluding

Remarks The spread of a disease in a population can depend on the age of the individuals of this population. It is thus signi cant to consider the age of the individuals more e ectively to ght in order to slow down this disease. Our study showed that the mathematical model of stochastic type is the best model one. We can thus use this model to predict the spread of a disease when it starts in a population of very few individuals. On the other hand, when the number of individuals becomes signi cant, the using of deterministic model is imperative. If our studie nd that the stochatic model converges toward the deterministic, it is signi cant to know the law of the determistic model. This preocupation will be the odjet of our next study.

Références

[1] E.Padoux :Mathematical Models of Epidemics,CIMPA School Ziguinchor,Dec. 2015.

[2] E.Pardoux : Processus de Markov et applications. edition,2006.

[3] Hao Kang, Shigui Ruan, Yu : Age-Structured Population Dynamics with Nonlocal Di usion Journal of Dynamics and Di erential Equations,Springer Science+Business Media, LLC, part of Springer Nature 2020

[4] Hisashi Inaba : Age-Structured Population Dynamics in Demography and Epidemiology The University of Tokyo Tokyo Japan, Springer Science+Business Media Singapore 2017 DOI 10.1007/978-981-10-0188-8

[5] H.Inaba :Thresold and stability results for an age-structured epidemic model, Journal of mathematical Biology, Springer-Verlag 1990.

[6] Jean-François LE GALL : MOUVEMENT BROWNIEN, MARTINGALES ET CALCUL STOCHASTIQUE. Université Paris-Sud 30 septembre 2011.

[7] Murel Casalis and Gérard letac : Multivariate quadratic natural exponabtial families Laboratoire de Statistique et Probabilité, Université de Paul Sabatier, 31062 Toulouse,France.

[8] Régis Ferrière.Viet Chi Tran : Stochastic and deterministic models for age-structured population with genetically variable traits. ESAIM : Proceedings, EDP Science,2009,27,pp.289- 310.

[9] Thierry Lubin : Equations aux dérivées partielles (EDP), Méthode de résolution des EDP par séparation de variables ; Applications HAL Id : Submitted on 21 Aug 2017

[10] Zhihan Feng, Wenzhang Huang, Carlos Castillo-Chavez : Global behavior of a multi-group SIS epidemic model with age structure Elsevier, Di rential Equations 218 (2005) 292-324.