

An Analytical Model for Concentration of Water Pollutant

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Abstract:

In this paper, an advection- dispersion equation for the water pollutant concentration is solved assuming zero initial rate of pollutant. To solve the dispersion equation in unsteady state condition, the Laplace transformation technique has been used. It is obtained that the concentration of the pollutants in a small river decreases continuously with increasing distance.

Keywords: Concentration of pollutant, Laplace transforms method, Advection – dispersion, Unsteady-state.

Introduction:

There is a big concern on the environment with increment of different pollutants. Water pollution is one of the serious problems for our society and thus for human beings. Several analytical models have been given to study for water pollutants concentration using dispersion process under different conditions.

Different models with different ideology have been used for the prediction of concentration of pollutants. Water pollutants are biological waste [8]. Jaiswal et al. obtained an analytical solution of the dispersion problem representing one coefficient dispersion parameter and the other representing velocity of fluid [3]. Van Genuchten solved a dispersion problem considering dispersion along unsteady flow [14].

Studies in this direction have given by Savovic, S. and Djordjevich [12,13], Mourad et al.[7], Pimpunchat et al. [8,9], Chrysikopoulos, C.V. and Sim, Y. [2], Sirin, H.[13], kumar A. et al.[5,6], Aral, M.M. and Liao, B. [1], Savovic,S.and Caldwell, J. [10] and many others including Kim Kue et al. [4].

The objective of this paper is to solve the advection- diffusion equation in a finite domain by applying the Laplace transform technique and observe the concentration profile with zero initial rate of pollutant along the river.

Mathematical formulation:

An unsteady flow of water pollutant concentration in one- dimension can be described by a partial differential equation as Pimpunch et al. [9]:

$$\frac{\partial(AC)}{\partial t} = D_x \frac{\partial^2(AC)}{\partial t^2} - \frac{\partial(UAC)}{\partial x} - k_1 \frac{x}{x+k} AC + q; \quad 0 \leq x < L, \quad t > 0 \quad (1)$$

where U is the water velocity in x - direction and described by $U = U_0(1 + ax)$; U_0 is velocity at origin of the medium and $b = real\ constant = aL$ at $x = L$, C is the concentration of pollutant, D_x is the dispersion coefficient of pollutant in x - direction, k_1 is the degradation rate coefficient of pollutant, q is the added pollutant rate along the river, k is the half saturated oxygen demand concentration for pollutant decay, X is the concentration of the dissolved oxygen within the river and A is the cross-section of area of river.

We assume a small river which is considered to be homogeneous system and we take the parameters A, U, q, D_x, k_1 as constants over time and space. We take $k = 0$.

For much greater pollutants, D_x is approximately zero and so we take $D_x = 0$. Applying above conditions, the equation (1) becomes:

$$\frac{\partial(C)}{\partial t} = -\frac{\partial(UC)}{\partial x} - k_1C + \frac{q}{A}; \quad 0 \leq x < L, \quad t > 0$$

$$\text{or } \frac{\partial(C)}{\partial t} = -U\frac{\partial C}{\partial x} - k_1C - U_0aC + \frac{q}{A}; \quad (2)$$

Now, equation (2) is solved under the following conditions:

$$C(x, 0) = 0; \quad x \geq 0 \quad (3)$$

$$C(0, t) = p; \quad t > 0 \quad (4)$$

where $C(x, t)$ is the pollutant concentration for the case when dispersion coefficient $D_x = 0$, the initial rate of pollution along the river is supposed zero and p is the rate of pollution at the origin.

We use Laplace transformation technique to solve equation (2) subject to initial condition (3) and boundary condition (4).

Thus, applying Laplace transformation to (2) and (4), we have

$$s\bar{C}(x, s) - C(x, 0) = -U\frac{\partial\bar{C}(x, s)}{\partial x} - k_1\bar{C}(x, s) - U_0a\bar{C}(x, s) + \frac{q}{sA}$$

$$\text{or } s\bar{C}(x, s) - C(x, 0) = -U\frac{\partial\bar{C}(x, s)}{\partial x} - (k_1 + U_0a)\bar{C}(x, s) + \frac{q}{sA} \quad (5)$$

$$\text{and } \bar{C}(0, s) = \frac{p}{s} \quad (6)$$

where s is the Laplace transform variable, and $\bar{(-)}$ denotes the corresponding Laplace transform of the function.

By using (3), equation (5) can be re-written as follows:

$$s\bar{C}(x, s) = -U\frac{\partial\bar{C}(x, s)}{\partial x} - (k_1 + U_0a)\bar{C}(x, s) + \frac{q}{sA};$$

which on simplification and using $U = U_0(1 + ax)$ gives

$$\frac{\partial\bar{C}(x, s)}{\partial x} + \left(\frac{s+k_1+U_0a}{U_0(1+ax)}\right)\bar{C}(x, s) = \left(\frac{q}{sA}\right)\frac{1}{U_0(1+ax)} \quad (7)$$

To solve equation (7), we find $I.F. = e^{\int \frac{s+k_1+U_0a}{U_0(1+ax)} dx} = (1 + ax)^{\frac{s+k_1+U_0a}{U_0a}}$

Thus, the solution of equation (7) is given by

$$\bar{C}(x, s)(1 + ax)^{\frac{s+k_1+U_0a}{U_0a}} = \int \left(\frac{q}{sA}\right)\frac{1}{U_0(1+ax)} \cdot (1 + ax)^{\frac{s+k_1+U_0a}{U_0a}} dx$$

which on simplification gives

$$\bar{C}(x, s) = \left(\frac{q}{As}\right) \frac{1}{s+k_1+U_0a} + c_1(1+ax)^{-\frac{s+k_1+U_0a}{U_0a}} \quad (8)$$

where c_1 is an arbitrary constant.

Now, applying the condition (6) to equation (8), we get

$$\frac{p}{s} = \left(\frac{q}{As}\right) \frac{1}{s+k_1+U_0a} + c_1 \quad \text{which gives} \quad c_1 = \left\{ \frac{p}{s} - \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} \right\}$$

Using this value of c_1 in equation (8), we get

$$\bar{C}(x, s) = \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} + \left\{ \frac{p}{s} - \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} \right\} (1+ax)^{-\frac{s+k_1+U_0a}{U_0a}}$$

$$\text{or } \bar{C}(x, s) = \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} + \left\{ \frac{p}{s} - \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} \right\} (1+ax)^{-\frac{s+k_1+U_0a}{U_0a}}$$

$$\text{or } \bar{C}(x, s) = \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} + \left\{ \frac{p}{s} - \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} \right\} (1+ax)^{-\frac{s+k_1+U_0a}{U_0a}}$$

Thus, we have

$$\bar{C}(x, s) = \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} + \frac{p}{s} (1+ax)^{-\frac{s+k_1+U_0a}{U_0a}} - \left(\frac{q}{sA}\right) \frac{1}{(s+k_1+U_0a)} (1+ax)^{-\frac{s+k_1+U_0a}{U_0a}} \quad (9)$$

Taking inverse Laplace transform of equation (9), we have

$$\begin{aligned} C(x, t) &= \frac{q}{A} \left(\frac{1}{(k_1+U_0a)} - \frac{1}{(k_1+U_0a)} e^{-(k_1+U_0a)t} \right) + p \left[(1+ax)^{-\frac{k_1+U_0a}{U_0a}} \cdot H \left(t + \frac{\log(1+ax)}{aU_0} \right) \right] \\ &\quad - \frac{q}{A} \left(\frac{1}{(k_1+U_0a)} - \frac{1}{(k_1+U_0a)} e^{-(k_1+U_0a)t} \right) \left[(1+ax)^{-\frac{k_1+U_0a}{U_0a}} \cdot H \left(t + \frac{\log(1+ax)}{aU_0} \right) \right] \end{aligned} \quad (10)$$

where $H \left(t + \frac{\log(1+ax)}{aU_0} \right)$ is Heaviside function.

For $t > -\frac{\log(1+ax)}{aU_0}$, equation (10) reduces to

$$\begin{aligned} C(x, t) &= \frac{q}{A} \left(\frac{1}{(k_1+U_0a)} - \frac{1}{(k_1+U_0a)} e^{-(k_1+U_0a)t} \right) + p \left[(1+ax)^{-\frac{k_1+U_0a}{U_0a}} \right] \\ &\quad - \frac{q}{A} \left(\frac{1}{(k_1+U_0a)} - \frac{1}{(k_1+U_0a)} e^{-(k_1+U_0a)t} \right) \left[(1+ax)^{-\frac{k_1+U_0a}{U_0a}} \right] \\ \text{or } C(x, t) &= \frac{q}{A} \cdot \frac{1}{(k_1+U_0a)} - \frac{q}{A} \cdot \frac{1}{(k_1+U_0a)} e^{-(k_1+U_0a)t} + p(1+ax)^{-\frac{k_1+U_0a}{U_0a}} \\ &\quad - \frac{q}{A} \cdot \frac{1}{(k_1+U_0a)} (1+ax)^{-\frac{k_1+U_0a}{U_0a}} + \frac{q}{A} \cdot \frac{1}{(k_1+U_0a)} e^{-(k_1+U_0a)t} (1+ax)^{-\frac{k_1+U_0a}{U_0a}} \end{aligned} \quad (11)$$

Now, using the dimensionless quantities: $x' = ax$, $t' = (k_1 + aU_0)t$, $p' = \frac{p}{\left(\frac{q}{(k_1 + aU_0)A}\right)}$, $C'(x', t') = \frac{C(x,t)}{\left(\frac{q}{(k_1 + aU_0)A}\right)}$, $k_1' = \frac{k_1}{aU_0}$, equation (11) becomes:

$$C'(x', t') = 1 - e^{-t'} + (1 + x')^{-(k_1'+1)} + e^{-t'} (1 + x')^{-(k_1'+1)} + p' (1 + x')^{-(k_1'+1)} \tag{12}$$

Results and Discussion:

The concentration $C'(x', t')$ given by (12) is in non-dimensional form. The parametric values used in the equation for finding the concentration profile are taken as Pimpunchat et al. [8].

$$t' = 0.937(t = 0.1 \text{ day}), 3.352(t = 0.4 \text{ day}), 5.86(t = 0.7 \text{ day}),$$

$$q = 0.06 \text{ kg/m}, \quad k_1 8.27 \text{ per day}, \quad A = 2100 \text{ m}^2, \quad a = 1,$$

$$U_0 = 0.11 \quad p' = 29.33, \quad 58.66, \quad 87.99$$

To find the behavior of the concentration profile, we have shown the non dimensional concentration distribution for different conditions.

Figure (1) represents the concentration profile against the distance ($0 \leq x' \leq 1$) for constant value of time $t' = 0.937(t = 0.1 \text{ day})$, and different velocity p' at the origin. It is seen that as x' increases the value of $C'(x, t)$ decreases. It reaches a constant value near the sink. The effect of time is dominant near the upstream and very small near the downstream.

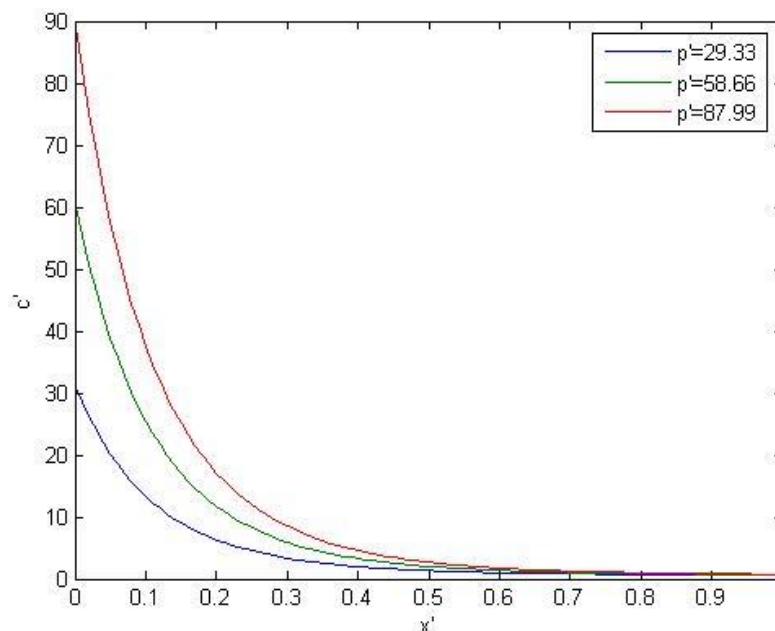


Fig1. Variation of concentration for different p' with constant t' .

Figure 2 represents the concentration profile against the distance ($0 \leq x' \leq 1$) for different value of t' and constant p' . As x' increases, the value of $C'(x, t)$ decreases for any time. Minimum value of $C'(x, t)$ is seen near the downstream. The effect of time is dominant near the upstream and very small near the downstream.

As velocity at the origin and time increases, the value of concentration increases at any cross section of the river.

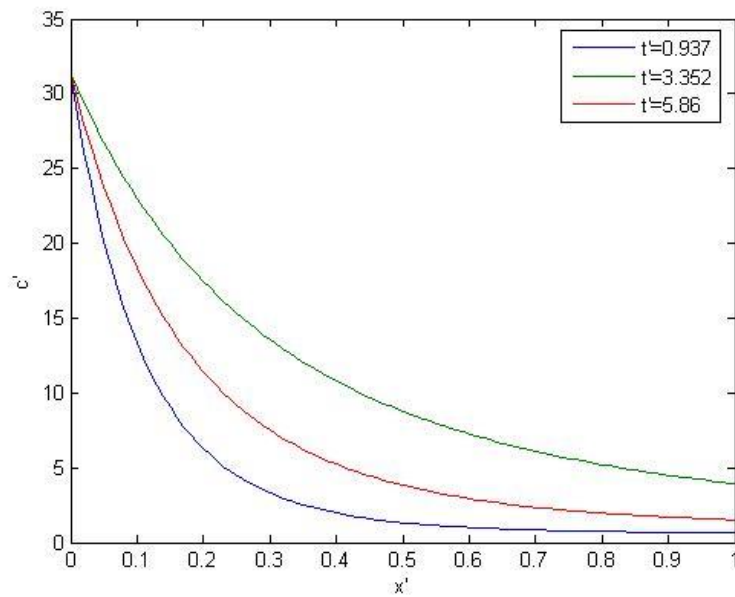


Fig2. Variation of concentration for constant p' and different t' .

Figure 3 represents the concentration profile against the distance ($0 \leq x' \leq 1$) for increasing value of time t' and decreasing the value of p' . It is found that the concentration $C'(x, t)$ is decreases as x' increases.

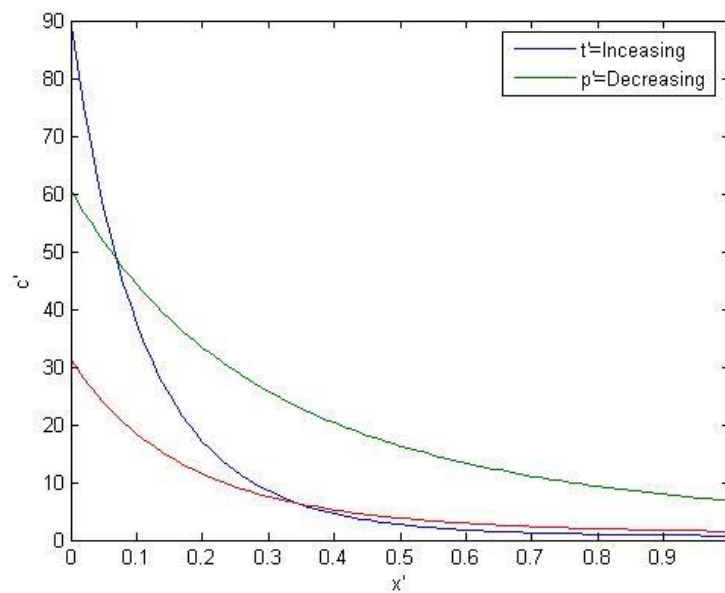


Fig3. Variation of concentration for increasing t' and decreasing p'

Conclusion:

It is observed that the concentration profile of pollutants is high near the source and as the distance from source increases, the concentration of pollutants decreases regularly. This study is applicable for the removal mechanism to decrease pollutants concentration from river water.

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