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Coupling of Laplace Differential Transform method with Padé Approximant for the Numerical solution of Initial and Boundary value problems

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Abstract:
This paper presents a study using a novel linearization technique based on the Differential transformation method (DTM) to seek analytical solutions if it exists and approximate solutions where closed form solutions are not available. The effectiveness and accuracy of this procedure is verified by solving six problems comprising both initial and boundary value problems by a combination of DTM and Laplace transform method. The resulting solution is then treated with Padé approximation to obtain a better approximation that converges to the exact solution. Simulated results of the study reveal the proposed technique is reliable, accurate and computationally convenient even with few iterations. The result obtained is in good agreement with existing literature.

Keywords: Differential Transform method (DTM), Laplace Transform method (LTM), Padé Approximation, Initial Value problems (IVP), Boundary value problems (BVP)

1. Introduction

Most phenomena of physical significance that abound in the field of mathematical sciences, social sciences and engineering are described in the form of mathematical models called differential equations where information about the initial and end points are given. These equations are called initial and boundary value problems. Plethora of methods ranging from approximate, semi-analytical and analytical have been proposed to solve these equations.

In practice, some of these equations have analytic or closed form solutions, whereas others don't have analytical solutions, hence academics resort to approximate or semi-analytical techniques for their solutions. Some of these methods includes: weighted residual methods (Galerkin, collocation, subdomain, Least-square, moment and Petrov-Galerkin), variational method (Rayleigh-Ritz), semi-analytical methods (Adomian decomposition method, Homotopy perturbation method, Homotopy analysis method, Abkari-Ganji's method, Variational Iteration method, Differential Quadrature method, Exp-Function method), Hybrid semi-analytical methods (Optimal Homotopy Asymptotic method, Spectral Homotopy Analysis method, Laplace Adomian decomposition method) and numerical methods (Finite Element method, Finite Volume method, Finite difference method) and Integral transform methods (Laplace and Fourier transformation).

The Differential Transformation method (DTM) was originally proposed by Zhou (1986) [1], while working on electrical circuits and its applications at the Huarjung University, Wuhan China. He successfully applied this novel technique to solve both linear and nonlinear initial value problems in circuit analysis. According to [2], DTM is an iterative technique used to obtain an analytic Taylor series solution to differential equation which converges to the exact solution. Usually, the Taylor series is computationally time consuming especially when the given problem involves higher orders. This method reduces the computational domain and reduce the problem to an algebraic equation, where solutions are obtained for different values of the iterates. In this method, certain transformation rules are applied to transform the given equation into an algebraic equation in terms of the differential transform of the original function, and the solution of the algebraic equation gives the desired solution of the problem. The main advantage of this method is that it can be applied directly to nonlinear differential equations without requiring discretization, linearization,

perturbation, small parameter, and auxiliary operator. For problems with a closed form solution, the approximate solution converges faster to the exact solution in finite iterative steps.
DTM coupled with Laplace transformation and Adomian decomposition method has been applied to solve diverse problems namely: excited nonlinear oscillator under the influence of damping, linear nonhomogeneous PDEs, ordinary differential equations, eigenvalue problems, IVPs for nonlinear differential equations, two-dimensional nonlinear Volterra integro-differential equations, nonlinear Emden-Fowler equations, seventh-order Sawada-Koterra equations, higher-order IVPs, linear and nonlinear high order BVPs, systems of differential equations, higher order linear BVPs, solution of thirteenth order BVPs, Delay-differential equations, linear and nonlinear wave equations, second order differential equations, linear and nonlinear systems of ordinary equations, nonlinear oscillators, nonlinear duffing oscillator with damping effects, Cauchy reaction-diffusion problems, one-space dimensional telegraph equation, nonlinear Volterra integro-differential equations, two singular BVPs, class of nonlinear singular BVPs, linear & nonlinear differential equations and systems of differential equations [3-36].
In this present article, we apply the DTM combined with Laplace transformation to solve initial and boundary value problems. The approximate solutions obtained from the model problem is then compared with the exact or analytical solution. In each case, the absolute error which is the norm between the exact and DTM solution is presented. The paper is organized as follows: In section 2, the fundamentals of the differential transformation technique is presented. The basic theorems or transformation rules of DTM are given in section 3. Numerical examples to illustrate the effectiveness, accuracy, and convergence of DTM is shown in section 4. In section 5 & 6, we draw the conclusion and discuss the results in tables and figures.

II. Basics of the Differential Transformation method (DTM)

Following the works of [2-3], we present the basics of the Differential transformation method
Let $u(t)$ be an analytic function in each domain D, differentiating $u(t)$ continuously kth times via Lebnitz theorem about $x = x_0$, where x_0 is the initial point of the function. The transformed function of $u(t)$ denoted $U(k)$ is given by

$$\frac{d^k u(t)}{dt^k} = \varphi(t,k),\ k\epsilon D \qquad (1)$$

Rewriting Eq. (1), we have the equivalent form as

$$U(k) = \varphi(t,k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0} \qquad (2)$$

Where $u(t)$ is the original function and $U(t)$ is the transformed or spectrum of $u(t)$.
Similarly, expressing $u(t)$ in Taylor series, the expression for the inverse transform of $u(t)$ becomes

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_0)^k}{k!}\right]U(k) \qquad (3)$$

Combining Eqs. (2) and (3) and using the differential operator, D as the transformation process

$$u(t) = \sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0} \qquad (4)$$

Using Eqs (2) and (4), the equivalent expression become

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_0)^k}{k!}\right]U(k) \equiv D^{-1}U(k) \qquad (5)$$

III. Special Theorems of Differential Transform Method (DTM)

Theorem 1. If $y(t) = u(t) \pm v(t)$, then $Y(k) = U(k) \pm V(k)$
Theorem 2. If $y(t) = \alpha u(t)$, then $Y(k) = \alpha U(k)$, where α is a constant
Theorem 3. If $y(t) = \frac{du(t)}{dt}$, then $Y(k) = (k+1)U(k+1)$
Theorem 4. If $y(t) = \frac{d^2u(t)}{dt^2}$, then $Y(k) = (k+1)(k+2)U(k+2)$
Theorem 5. If $y(t) = \frac{d^r u(t)}{dt^r}, then\ Y(k) = (k+1)(k+2)\ldots.(k+r)U(k+r)$
Theorem 6. If $y(t) = u(t)v(t)$, then $Y(k) = \sum_{n=0}^{k} U(k)V(k-n)$
Theorem 7. If $y(t) = t^r,\ Y(k) = \delta(k-r) = \begin{cases}1, & k=r\\ 0, & k\neq r\end{cases}$
Theorem 8. If $y(t) = u^3(t)$, then $Y(k) = \sum_{k_1}^{k}\sum_{r}^{k} U(r)U(k_1-r)U(k-r)$
Theorem 9. If $y(t) = (\frac{du(t)}{dt^2})^2$, then $Y(k) = \sum_{r}^{k}(k+1)(k-r+1)U(r+1)U(k-r+1)$

Theorem 10. If $y(t) = \frac{d^2u(t)}{dt^2}$, then $Y(k) = \sum_r^k (r+1)(r+2)(k-r+2)(k-r+1)U(r+2)U(k-r+2)$

Theorem 11. If $y(t) = e^{\lambda t}$, then $Y(k) = \frac{\lambda^k}{k!}$

Theorem 12. If $y(t) = (1+t)^r$, then $Y(k) = \frac{r(r-1).....(r-k+1)}{k!}$, then

Theorem 13. If $y(t) = \sin(nt+\alpha)$, then $Y(k) = \frac{n^k}{k!}\cos(\frac{\pi k}{2}+\alpha)$

Theorem 14. If $y(t) = \cos(nt+\alpha)$, then $Y(k) = \frac{n^k}{k!}\sin(\frac{\pi k}{2}+\alpha)$

IV. Padé Approximation

In the approximating functions in mathematics and other sciences, the Taylor series use repeated differentiation to produce a polynomial approximation about a particular point. However, this method has an obvious drawback, that it can't extrapolate a function for much long before diverging rapidly to positive or negative infinity. Hence, a new method of approximation that follow closely the function for much longer is needed.

Henri Pad**é** (1863-1953) presented an approximation technique in his doctoral thesis in 1892, which is an extension of the Taylors series polynomial approximation for rational functions. A Pad**é** approximant approximates a function using rational polynomials expressed in the form.

$$f(x) \sim \frac{a_0+a_1x+a_2x^2+\cdots+a_Nx^N}{b_0+b_1x+b_2x^2+\cdots+b_Mx^M} \tag{6}$$

Thus, an $[N/M]$ Pad**é** approximant is formed from an Nth degree polynomial in the numerator and an Mth degree polynomial in the denominator.

$$[N/M]P(x) = P_M^N(x) = \frac{\sum_{i=1}^N a_i x^i}{\sum_{j=0}^M b_j x^j} \tag{7}$$

The Pad**é** approximant is advantageous over the Taylor series in that it follows the function for much longer. It is constructed to agree with the first $N+M$ terms of the Taylor series.

Steps in Constructing the $[N/M]$ Padé Approximant

Step 1. Calculate the Taylor series of the given function

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{N+M}x^{(N+M)} \tag{8}$$

Step 2. Equate the Taylor series of the function in Eq. (6) to the Pad**é** approximant and apply the normalization condition by setting $b_0 = 1$. This is without loss of generality, to ensure easier calculation.

$$\frac{a_0+a_1x+a_2x^2+\cdots+a_Nx^N}{1+b_1x+b_2x^2+\cdots+b_Mx^M} \tag{9}$$

Step 3. Multiply both sides by the denominator and ignore any term longer than $(N+M)$. This gives the equivalent form.

$$\sum_{n=0}^{\infty} c_n x^n = \frac{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_Nx^N}{1+b_1x+b_2x^2+b_3x^3+\cdots+b_Mx^M} + O(x^{N+M+1}) \tag{10}$$

Writing the above in explicit form, we get the form

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n - \left(\frac{a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_Nx^N}{1+b_1x+b_2x^2+b_3x^3+\cdots+b_Mx^M}\right) = O(x^{L+M+1}) \tag{11}$$

$$a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N = (1+b_1x+b_2x^2+\cdots+b_Mx^M)\left(c_0+c_1x+c_2x^2+\cdots+c_{N+M}x^{(N+M)}\right) \tag{12}$$

Step 4. Equating the terms in power of x gives a system of $(N+M)$ equations in $(N+M)$ unknowns.

$$\left.\begin{array}{c} a_0 = c_0 \\ c_1 + b_1 c_0 = a_1 \\ c_2 + b_1 c_1 + \cdots + b_0 c_0 = a_2 \\ \vdots \\ c_N + b_1 c_{N-1} + \cdots + b_N c_o = a_N \end{array}\right] N+1 \qquad (13)$$

and

$$\left.\begin{array}{c} c_{N+1} + c_N b_1 + \cdots + b_N c_2 = 0 \\ c_{N+2} + c_{N+1} b_1 + \cdots + b_N c_2 = 0 \\ \ddots \\ c_{N+M} + c_{2N-1} b_1 + \cdots + b_N c_N = 0 \end{array}\right] M-1 \qquad (14)$$

For $n < 0,$ we set $a_n = 0$ and $q_j = 0$ for $j > m$. Suppose the Eqs. (13) and (14) are non-singular, then solving indirectly gives the condition in determinant form as

$$[N/M] = \frac{\begin{vmatrix} c_{N-M+1} & c_{N-M+2}\cdots & c_{N+1} \\ c_N & c_{N+1}\cdots & c_{N+M} \\ \vdots & \vdots & \vdots \\ \sum_{j=M}^{L} c_{j-M} x^j & \sum_{j=M-1}^{L} c_{j-M+1} x^j & \sum_{j=0}^{L} c_j x^j \end{vmatrix}}{\begin{vmatrix} c_{N-M+1} & c_{N-M+2}\cdots & c_{N+1} \\ c_N & c_{N+1}\cdots & c_{N+M} \\ \vdots & \vdots & \vdots \\ x^M & x^{M-1} & 1 \end{vmatrix}} \qquad (15)$$

V. Computation and Results

In this subsection, we apply the combination of Laplace and Differential Transformation methods to six problems comprising of both initial and boundary values to demonstrate the workability and efficiency of the method. The obtained solution in series form is then treated with Padé approximation to obtain an approximate solution that converges to the exact solution.

Example 5.1 Consider the second order initial value problems

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 24 \qquad (16)$$

With initial conditions

$$y(0) = 10, y'(0) = 0 \qquad (17)$$

Applying DTM of both sides of the equation, we have

$$(k+2)!\,Y(k+2) + 3(k+1)!\,Y(k+1) + 2Y(k) = 24\delta(k)$$

$$(k+1)(k+2)Y(k+2) + 3(k+1)Y(k+1) + 2Y(k) = 24\delta(k)$$

Rearranging the above, the recurrence relation become

$$Y(k+2) = \frac{1}{(k+1)(k+2)}[-3(k+1)Y(k+1) - 2Y(k) + 24\delta(k)] \qquad (18)$$

Similarly, the initial conditions become

$$Y(0) = 10 \text{ and } Y(1) = 0 \qquad (19)$$

Using the recurrence relation in Eq. (18), subject to the initial condition in Eq. (19).

For $k = 0,1,2,3,4$ ...., we obtained the values $Y(0) = 10, Y(1) = 0, Y(2) = 2, Y(3) = -2, Y(4) = {}^7/_6$

The solution on using, $y(x) \approx \sum_{k=0}^{\infty} x^k Y(k)$ become

$$y(x) = 10 + 2x^2 - 2x^3 + \frac{7}{6}x^4 \qquad (20)$$

Taking the Laplace transform of Eq. (20), we obtain

$$\mathcal{L}\{y(x)\} = 10x + 4x^3 - 12x^4 + 28x^5 \qquad (21)$$

Next, we take the Pad**é** approximant of the above, replacing x by $1/s$ and taking the inverse Laplace transform

$$y(x) = 12 - 4e^{-x} + 2e^{-2x} \qquad (22)$$

Example 5.2 Consider the nonlinear IVP

$y''(x)+2\left(y'(x)\right)^2+8y(x)=0, 0\le x<\infty$ (23)
Subject to the initial conditions
$y(0)=0$ and $y'(0)=1$ (24)
Taking the DTM of both sides of Eq. (25), we obtain

$$(k+2)!\,Y(k+2)+2\sum_{k=0}^{k}(r+1)(k-r+1)Y(r+1)Y(k-r+1)+8Y(k)=0$$

$(k+1)(k+2)Y(k+2)+2\sum_{k=0}^{k}(r+1)(k-r+1)Y(r+1)Y(k-r+1)+8Y(k)=0$ (25)
Rearranging Eq. (25), the recurrence relation become
$Y(k+2)=\frac{-1}{(k+1)(k+2)}\left[2\sum_{k=0}^{k}(r+1)(k-r+1)Y(r+1)Y(k-r+1)+8Y(k)=0\right]$ (26)
The corresponding initial conditions become
$Y(0)=0$ and $Y(1)=1$ (27)
For $0\le k\le 4$, using the recurrence relation subject to the condition in Eq. (27), we obtained the functional values as

$$Y(0)=0, Y(1)=1, Y(2)=-1, Y(3)=0, Y(4)=0$$

Clearly, for $k\ge 3, Y(k)=0$, hence the solution become
$y(x)=x-x^2$ (28)
Taking the Laplace transform of both sides of Eq. (28), we get

$$\mathcal{L}\{y(x)\}=\frac{1}{s^2}-\frac{2}{s^3}$$

Replacing x by $1/s$ in the above, we obtain
$\mathcal{L}\{y(x)\}=x^2-2x^3$ (29)
Taking the Padé $[2/2]\left(y(x)\right)$ $a$pproximant and replacing x by $1/s$ gives the expression
$P_2^2(x)=\frac{1}{(1+\frac{4}{s^2}+\frac{2}{s})s^2}$ (30)
Taking the inverse Laplace transform of both sides gives the exact solution as
$y(x)=\frac{e^{-x}\sin[\sqrt{3}x]}{\sqrt{3}}$ (31)

Example 5.3 Consider the first order IVP
$y'-y+1=0$ (32)
With the initial conditions
$y(0)=2$ (33)
Taking the DTM to both sides of the equation
DTM$(y'-y+1=0)$

$$(k+1)!\,Y(k+1)-Y(k)+\delta(k)=0$$

$$(k+1)Y(k+1)=Y(k)-\delta(k)$$

$Y(k+1)=\frac{1}{k+1}[Y(k)-\delta(k)]$ (34)
The initial condition become
$Y(0)=2$ (35)
For $k\ge 0$, we obtain the values of $Y(1), Y(2), Y(3), Y(4), Y(5), Y(6), \ldots$as

$$Y(0)=2, Y(1)=1, Y(2)=\frac{1}{2}, Y(3)=\frac{1}{4!}, Y(4)=\frac{1}{5!}, Y(5)=\frac{1}{5!}, ..$$

Using the relation, $y(x)\approx\sum_{k=0}^{\infty}x^kY(k)$, we have the solution as
$y(x)=2+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$ (36)
Taking the Laplace transform of Eq. (35), we have
$\mathcal{L}\{y(x)\}=\frac{2}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}+\frac{1}{s^5}+\frac{1}{s^6}$ (37)
Replacing s by $1/x$, we obtain

$\mathcal{L}\{y(x)\} = 2x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$ (38)

Taking the Padé approximant and replacing x by $1/s$ in the resulting expression, we obtain the form

$Pade[3/3](y(x)) = \frac{\frac{2}{s} - \frac{1}{s^2}}{1 - \frac{1}{s}}$ (39)

Taking the inverse Laplace transform of both sides of Eq. (37), we get the exact solution as

$$y(x) = 1 + e^x$$

Example 5.4 Consider the IVP

$y'(x) - y^2(x) = 1$ (40)

With the initial condition

$y(0) = 0$ (41)

Taking the DTM of both sides of Eq. (39), we have

DTM$(y'(x) - y^2(x) = 1)$

$$(k+1)!\,Y(k+1) - \sum_{l=0}^{k} Y(l)Y(k-l) = \delta(k)$$

Rearranging the above gives the recurrence relation as

$Y(k+1) = \frac{1}{(k+1)!}\left[\delta(k) + \sum_{l=0}^{k} Y(l)Y(k-l)\right]$ (42)

The initial condition becomes

$Y(0) = 0$ (43)

For $k \geq 0$, we obtain the values of the functionals, $Y(1), Y(2), Y(3), Y(4), Y(5), Y(6), \ldots$ as

$$Y(1) = 1, Y(2) = 0, Y(3) = 1/3, Y(4) = 0, Y(5) = 2/15, Y(6) = 0$$

Clearly, for $k \geq 1, Y(0) = Y(2) = Y(4) \ldots Y(2k-2) = 0$

On using the relation, $y(x) \approx \sum_{k=0}^{\infty} x^k Y(k)$, the approximate solution become

$y(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$ (44)

Taking the Laplace transform of Eq. (44), we have the corresponding expression

$$\mathcal{L}\{y(x)\} = \frac{1}{s^2} + \frac{2}{s^4} + \frac{16}{s^6} + \cdots$$

$\mathcal{L}\{y(x)\} = \frac{1}{s^2} + \frac{2}{s^4} + \frac{16}{s^6} + \cdots$ (45)

Replacing s by $1/x$ in Eq. (6), we get

$\mathcal{L}\{y(x)\} = x^2 + 2x^4 + 16x^6$ (46)

Taking the Padé approximant, replacing x by $1/s$ and the inverse Laplace transform of both sides gives the exact solution as

$y(x) = \frac{e^{-\sqrt{2}x}(-1 + e^{2\sqrt{2}x})}{2\sqrt{2}}$ (47)

Example 5.5 Consider the initial value problem

$y'' = x + y - y^2$ (48)

Initial condition

$y(0) = -1$ and $y'(0) = 1$ (49)

Taking the DTM of both sides, we have the corresponding expression as follows

DTM(y'') =DTM$(x + y - y^2)$

$$(k+2)!\,Y(k+2) = \delta(k-1) + Y(k) - \sum_{r=0}^{k} Y(r)Y(k-r)$$

Rearranging, the recurrence relation become

$Y(k+2) = \frac{1}{(k+1)(k+2)}[\delta(k-1) + Y(k) - \sum_{r=0}^{k} Y(r)Y(k-r)]$ (50)

For $k = 0,1,2,3,4,5,..$ we obtain the values of the functions, $Y(1), Y(2), Y(3), Y(4), Y(5)$ as follows

$$Y(0) = -1, Y(1) = 1, Y(2) = -1, Y(3) = 2/3\,, Y(4) = -1/3\,, Y(5) = 1/5$$

Substituting the functional values into the relation, $y(x) \approx \sum_{k=0}^{\infty} x^k Y(k)$, the solution become

$$y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 - \cdots \quad (51)$$

Taking the Laplace transform of both sides of Eq. (52), we have the expression

$$\mathcal{L}\{y(x)\} = -\frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^3} + \frac{4}{s^4} - \frac{8}{s^5} + \frac{24}{s^6} \quad (52)$$

Replacing s by $1/x$, we get

$$\mathcal{L}\{y(x)\} = -x + x^2 - 2x^3 + 4x^4 - 8x^5 + 24x^6$$

Taking the Padé approximant, replacing x by $1/s$, we get the expression as follows

$$P_3^3(x) = \frac{-1+\frac{2}{3s^3}-\frac{3}{5s^2}+\frac{2}{s}}{1-\frac{2}{5s^3}-\frac{7}{5s^2}-\frac{1}{s}} \quad (53)$$

Taking the inverse Laplace transform of both sides, we obtain the exact solution of the form

$$y(x) = x - \cos x \quad (54)$$

Example 5.6 Solve the boundary value problem

$$y''(x) - 16y'(x) + 64y = 0 \quad (55)$$

With the initial condition

$$y(0) = 3 \text{ and } y(1) = 7 \quad (56)$$

Taking the DTM of both sides of Eq. (55) subject to the boundary condition, we have

$\text{DTM}(y''(x) - 16y'(x) + 64y = 0)$

$$(k+2)!\,Y(k+2) - 16(k+1)!\,Y(k+1) + 64Y(k) = 0$$

$$(k+1)(k+2)Y(k+2) - 16(k+1)Y(k+1) + 64Y(k) = 0$$

Rearranging the above, we get the equivalent expression as follows

$$Y(k+2) = \frac{1}{(k+1)(k+2)}[16(k+1)!\,Y(k+1) - 64Y(k)] \quad (57)$$

For $k \geq 2$, we obtain the values of the function, $Y(2), Y(3), Y(4), Y(5), Y(6), \ldots$ as follows

$$Y(2) = -40, Y(3) = \frac{16}{3}(-54)$$

Using the relation, $y(x) \approx \sum_{k=0}^{\infty} x^k Y(k)$, the approximate solution of the problem become

$$y(x) = 3 + 7x - 40x^2 - 288x^3 \quad (58)$$

Taking the Laplace transform of both sides of Eq. (58) gives

$$\mathcal{L}\{y(x)\} = \frac{3}{s} + \frac{7}{s^2} - \frac{80}{s^3} + \frac{4}{s^4} - \frac{8}{s^5} + \frac{24}{s^6} \quad (59)$$

Replacing s by $1/x$, we have the following

$$\mathcal{L}\{y(x)\} = 3x + 7x^2 - 80x^3 - \cdots \quad (60)$$

Taking the $[2/2]$ Padé approximant and replacing x by $1/s$, we get an expression of the form

$$P_2^2(x) = \frac{\frac{3703}{289s^2}+\frac{3}{s}}{1+\frac{6400}{289s^2}+\frac{560}{289s}} \quad (61)$$

Taking the inverse Laplace transform of both sides of Eq. (60), we obtain the exact solution as

$$y(x) = \frac{e^{-280t/289}(360\sqrt{123}\text{Cos}[\frac{120\sqrt{123}t}{289}]+2863\text{Sin}[\frac{120\sqrt{123}t}{289}])}{120\sqrt{123}} \quad (62)$$

VI. Numerical Results.

In this subsection, we present the results of the considered problems graphically and in tables. The comparison of the exact and approximate solution obtained as well as the absolute error is tabulated.

Table 6.1 Computation of Absolute Error, DTM Method and Exact Solutions for Example 1 with $h = 1$ using five iterates.

x	Exact Solution $y(x)$	DTM Method	Absolute Error= $\|\|y(x) - \hat{y}(x)\|\|$
0	10.00000	10.00000	0.00000

1	10.79920	11.16670	0.36750
2	11.49530	20.66670	9.17140
3	11.80530	68.50000	56.69420
4	11.92740	212.6670	200.7390
5	11.97310	539.1670	527.1940
6	11.99010	1162.0000	1150.0100
7	11.99640	2223.1700	2211.1700
8	11.99870	3892.6700	3880.6700
9	11.99950	6368.5000	6356.5000
10	11.99980	9876.6700	9864.6700

Table 6.2 Computation of Absolute Error, DTM Method and Exact Solutions for Example 2 with $h = 1$ using five iterates.

x	Exact Solution $y(x)$	DTM Method	Absolute Error= $\|y(x) - \hat{y}(x)\|$
0	0.00000	0.00000	0.00000
1	0.20964	0.00000	0.73502
2	-0.02477	-2.00000	18.34280
3	-.0.02545	-6.0000	113.38800
4	0.00636	-12.00000	401.47900
5	0.00269	-20.00000	1054.39000
6	-0.00118	-30.0000	2300.02000
7	-0.00023	-42.00000	4422.3400
8	-0.00019	-56.00000	7761.3400
9	8.49464E-6	-72.00000	12713.0000
10	-0.000260	-90.00000	19729.3000

Table 6.3 Computation of Absolute Error, DTM Method and Exact Solutions for Example 3 with $h = 1$ using five iterates.

x	Exact Solution $y(x)$	DTM Method	Absolute Error= $\|y(x) - \hat{y}(x)\|$
0	2.00000	2.00000	0.00000
1	3.71828	3.71667	1.10254
2	8.38906	8.26667	27.51410
3	21.08550	19.40000	170.08300
4	55.59820	43.86670	602.21800
5	149.41300	92.41670	1581.58000
6	404.42900	180.80000	3450.0300
7	1097.63000	330.76700	6633.51000
8	2981.96000	571.06700	11642.0000
9	8104.08000	938.45000	19069.5000
10	22027.500	1478.6700	29594.0000

Table 6.4 Computation of Absolute Error, DTM Method and Exact Solutions for Example 4 with $h = 1$ using five iterates.

x	Exact Solution $y(x)$	DTM Method	Absolute Error= $\|y(x) - \hat{y}(x)\|$
0	0.00000	0.00000	0.000000
1	1.46667	0.35350	1.477006
2	8.93333	0.35355	36.68550
3	44.40000	0.24312	226.77700
4	161.86700	0.353550	802.9570
5	463.33333	0.003494	2108.770
6	1114.800	0.35355	4600.0400
7	2362.2700	0.41625	8844.6800
8	4547.7300	0.432760	15522.700
9	8125.2000	0.762700	25426.000
10	13676.700	0.792900	39458.700

Table 6.5 Computation of Absolute Error, DTM Method and Exact Solutions for Example 5 with $h = 1$ using five iterates.

x	Exact Solution $y(x)$	DTM Method	Absolute Error= $\|y(x) - \hat{y}(x)\|$
0	-1.00000	-1.00000	0.00000
1	-0.46680	-0.46667	0.00013
2	3.42000	3.40000	0.02000
3	32.61000	32.60000	0.01000
4	149.14200	149.1330	0.00900
5	479.0000	478.0000	0.00100
6	1240.2000	1236.2000	4.00000
7	2746.7400	2746.7300	0.01000
8	5473.6000	5472.6000	1.00000
9	10035.900	10035.800	0.10000
10	17243.3000	17242.300	1.00000

Table 6.6 Computation of Absolute Error, DTM Method and Exact Solutions for Example 6 with $h = 1$ using five iterates.

x	Exact Solution $y(x)$	DTM Method	Absolute Error= $\|y(x) - \hat{y}(x)\|$
0	0.00000	0.00000	0.00000
1	-2.46020	-70.0000	2.20508
2	-2.47295	-606.0000	55.02830
3	2.98999	-2088.000	340.1650
4	1.83290	-4996.000	1204.4400
5	-3.38255	-9810.000	3163.1600
6	-1.10774	-17010.000	6900.060
7	3.619860	-27076.000	13267.000
8	0.332250	-40488.000	23284.000
9	-3.69104.00	-57726.000	38139.000
10	0.458450	-79270.000	59188.000

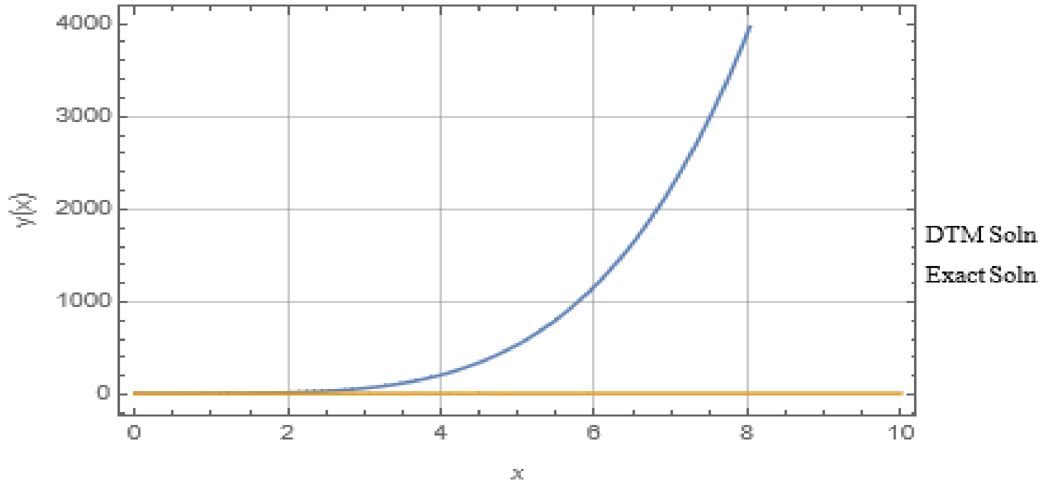

Figure 6.1. Variation of DTM and Exact solution for the first IVP

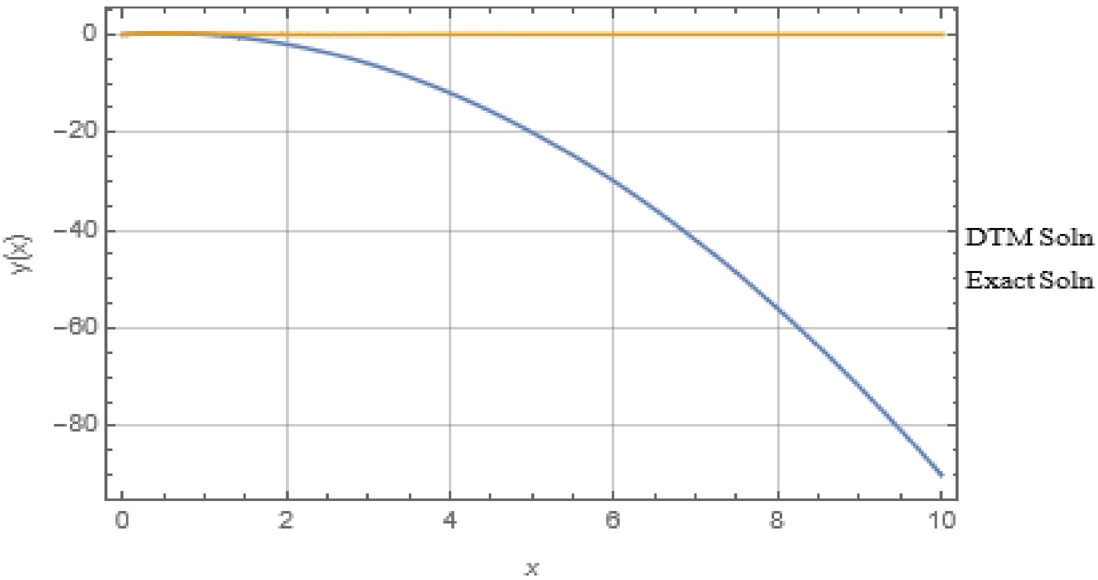

Figure 6.2. Variation of DTM and Exact solution for the first IVP

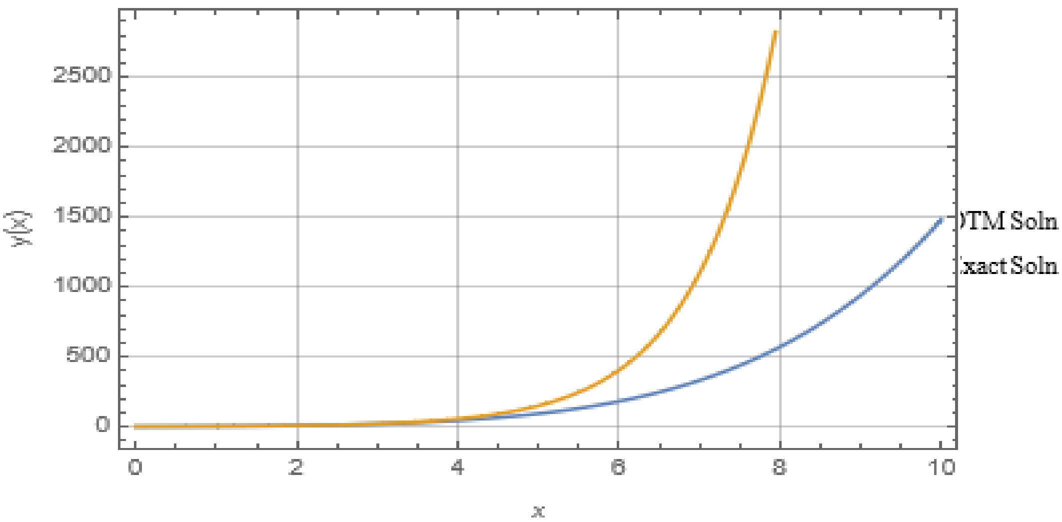

Figure 6.3. Variation of DTM and Exact solution for the first IVP

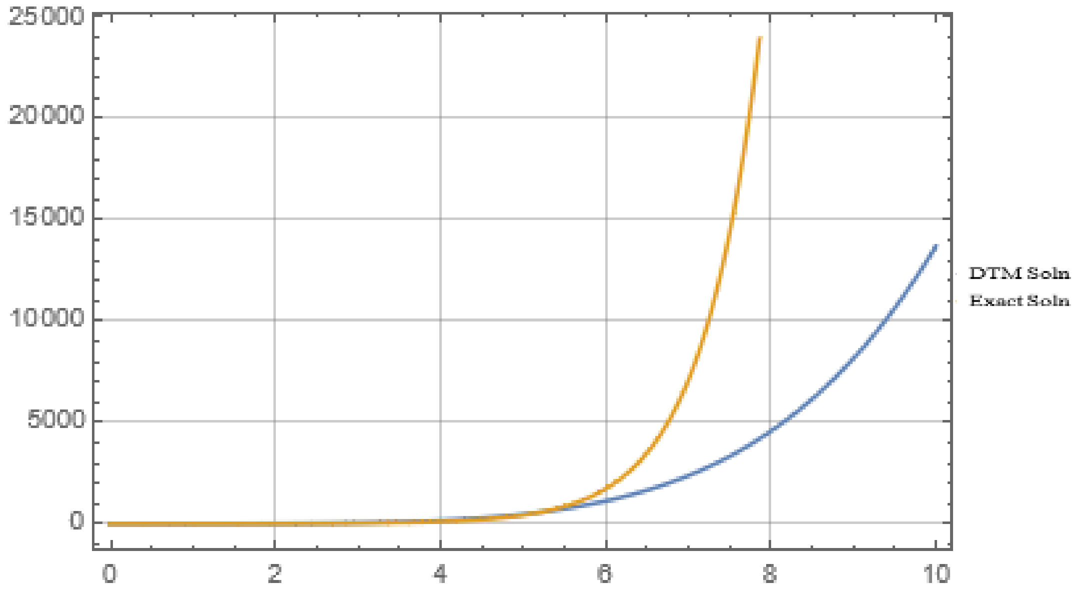

Figure 6.4. Variation of DTM and Exact solution for the first IVP

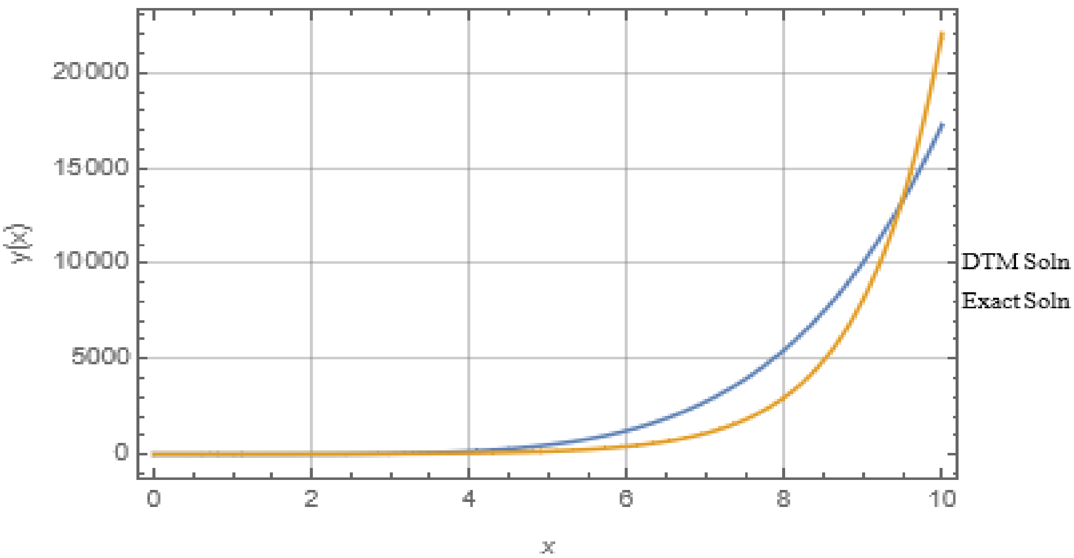

Figure 6.5. Variation of DTM and Exact solution for the first IVP

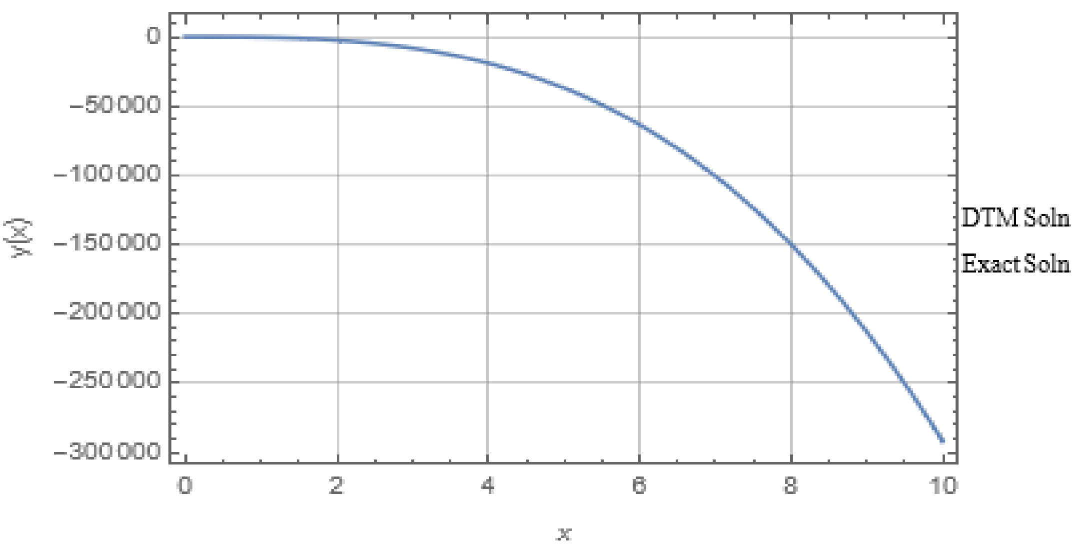

Figure 6.6. Variation of DTM and Exact solution for the first IVP

VII. Discussion of Results

In this research article, the combined treatment of Laplace and Differential Transform methods are coupled and applied to solve a class of initial and boundary value problems. To verify the efficiency, accuracy, and reliability of the proposed methods, six selected problems are successfully solved, and comparison is made between the exact solution and approximate solutions due to LDTM-Pad**é** and the resulting absolute error is ascertained. For problems with known analytical solution, the approximate solution converges to the exact solution in few iterative steps, whereas for others without closed form solution, only approximate solutions with minimal error were obtained. The solutions obtained showed the method is computationally efficient and mathematically elegant. The computations of the study were carried out using Maple 20, and the numerical results are presented in tables and figures.

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