Economic Order Model (Q,R): Constant Lead Times And Exponential Backorder Costs

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ABSTRACT

The paper considers the simple model of inventory models (Q,R) The backorder cost C_β/t) is taken as an exponential function of t, the length of time of the backorder, C_β(t) is $b_1 e^{b_2 t}$. The expected backorder cost is derived by obtaining the difference between the expected backorder cost at time t+L and t+L+T. In this paper demand is assumed to follow a normal distribution.Some basic mathematics of the properties of a normal distribution is introduced to simplify the derivation of the equations.
The first order derivatives of the inventory backorder costs are given.

INTRODUCTION

In this paper the cost depends upon the length of time for which the backorder exists and is taken as an exponential function. Backorders that are not met on time incur various types of costs, which could be linear, quadratic, exponential or any other function of the time orders are not met. Organizations that store thousands of products could face severe inventory costs, when the backorder cost is an exponential function.

Literature Review

This simple economic model (Q,R) is well dealt with by Hadley and Within (Ref 1) the linear backorder cost was considered.
Uthayakumar and Parrathi (2) investigate a continuous review inventory model to reduce lead time, yield variability and set up costs simultaneously, through capital investments. Zhang G.U and Dathwo (3) a hybrid inventory system with a time limit in backorders.

ECONOMIC ORDER MODEL (Q,R) CONSTANT LEAD TIME AND EXPONENTIAL COST TERMS

We will derive the model for the inventory backorder model when the backorder cost is an exponential function
$b_1 \exp\ (b_2 t)$ *where* $b_2 > 0$ *and t is the* for which the backorder exists.
The length of time for which the backorder exists directly would be used to derive the expected backorder costs.
Let C_β (t) be the cost of a backorder which last for time t

$$C_\beta(t) = b_1 \exp\ b_2 t \ b_2 > 0$$

Let g(x_1DL) be the probability density function of demand in a lead time L. x follows a normal distribution with mean DL and variance $\sigma^2 L$

$$g(x, DL) = esp - \tfrac{1}{2}\left(\frac{x - DL}{\sqrt{\sigma^2 L}}\right)^2 \quad -\alpha < x < \alpha$$

$$and\ F(R, DL) = \int_R^{\alpha} g(x, DL)\, dx$$

$$and\ R\, be\, the\, reorder\ level$$

If a backorder is incurred at time Z, Z < L then L-Z is the time for which the backorder lasts.

Let R+y +0 < y < Q be the inventory level at time 0, then if the system is out of stock in the time interval Z to Z + dz after the reorder point R is reached then R +y was demanded in time z and a demand occurred in time dz.

This probability is D.dz $\dfrac{1}{\sqrt{2\pi\sigma^2 L}} \exp - \dfrac{1}{2}\left(\dfrac{R - y - DZ}{\sqrt{\sigma^2 L}}\right)^2$ (1)

Hence the probability that
t = L-z, the length of time of a backorder

$$= \frac{D}{\sqrt{2\pi\sigma^2 L}} \exp \frac{-1}{2}\left(\frac{R + y - DZ}{\sqrt{\sigma^2 L}}\right)^2 dz,\ 0 < z\ L$$

Giving an inventory level R + y at time 0 expected cost of backorder

$= C_\beta(L - z)$ Probability of there been a backorder lasting L-z

$$= \frac{DC_\beta(L - z)}{\sqrt{2\pi\sigma^2 L}} \exp \frac{-1}{2}\left(\frac{R + y - DZ}{\sqrt{\sigma^2 L}}\right)^2 dz$$

Hence the expected backorder cost per cycle

$$= D\int_0^Q \int_0^L \frac{C_\beta(L - z)}{\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R + y - Dz}{\sqrt{\sigma^2 L}}\right)^2 dzdy \qquad (2)$$

Where R+y, 0 < y < Q , < Q is the inventory position at time 0, and $\sqrt{\sigma^2 L}$ is the standard deviation of demand over the lead time.

Form 1.

$$C_\beta(L - z) = b_1 esp\ b_2(L - z) \qquad (3)$$

Substituting into 2

We have $G_1(Q,R)$ expected backorder cost per cycle

$$G_1(Q,R) = D\int_0^Q \int_0^L \frac{b_1 e^{b_2(L - Z)}}{\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R + Y - Dz}{\sqrt{\sigma^2 L}}\right)^2 dzdy$$

Simplifying and noting that

$$\left(b_2(L - z) - (R + y - Dz)\right)^2 / 2\sigma^2 L$$

$$= b_2 L - \frac{1}{2\sigma^2 L}\left((R + y)^2 - 2Dz(R + y)D^2 z^2 + 2\sigma^2 b_2 zL \right)$$

Simplifying

$$then\ G_1(Q,R) = \frac{Db_1 e^{b_2 L}}{\sqrt{2\pi\,\sigma^2 L}} \int_0^Q \int_0^L \exp \frac{-1}{\sqrt{2\sigma^2 L}}\left((R + y)^2 - 2Dz(R + y) + D^2 z^2 + 2\sigma^2 L b_2 z \right) dzdy$$

$$G_1(Q,R)=\frac{Db_1e^{b_2L}}{\sqrt{2\pi\sigma^2L}}\int_0^Q\int_0^L \exp\frac{-1}{2\sigma^2L}\left[\left(\left(R+Y-\frac{\sigma^2b_2L}{D}\right)^2-2Dz\left(R+y-\frac{\sigma^2b_2L}{D}\right)+D^2z^2\right.\right.$$

$$\left.\left.-\left(\frac{\sigma^2b_2L^2}{D^2}-2\sigma^2L\frac{b_2(R+y)}{D}\right)\right)\right]dzdy$$

$$G_1(Q,R)=\frac{Db_1e^{b_2}L}{\sqrt{\sigma^2L}}\int_0^Q\exp\left(\frac{\sigma^2b_2^2L}{2D^2}-b_2\frac{(R+y)}{D}\right)\int_0^L \exp\frac{-1}{2}\left(\frac{R+y-\frac{\sigma^2b_2L}{D}-Dz}{\sqrt{\sigma^2L}}\right)^2 dzdy \qquad (4)$$

Let V = $\dfrac{R-y-\frac{\sigma^2b_2L}{D}-Dz}{\sqrt{\sigma^2L}}$

Thus d V = $\dfrac{-D}{\sqrt{\sigma^2}L}dz$

Substitute V into (4)

$$G_1(Q,R)=-b_1e^{b_2L}\int_0^Q \exp\left(\frac{\sigma^2b_2^2L}{2D^2}-\frac{b_2(R+y)}{D}\right)\int_{\left(R+y-\sigma^2Lb_2\right)/\sqrt{\sigma^2L}}^{\left(R+y-\frac{\sigma^2Lb_2}{D}-DL\right)/\sqrt{\sigma^2L}} g(v)dvdvdy$$

Integrating with respect to V

$$G_1(Q,R)=-b_1e^{b_2L}\int_0^Q\exp\left(\frac{\sigma^2b_2L}{D}-b_2\frac{(R+Y)}{D}\right)$$

$$*\left[F\left(\frac{R+y-\sigma^2Lb_2/D}{\sqrt{\sigma^2L}}\right)-F\left(\frac{R+Y-\frac{\sigma^2b_2L}{D}-DL}{\sqrt{\sigma^2L}}\right)\right]dy$$

Let $F\left(\dfrac{R+y-\sigma^2b_2L/D}{\sqrt{\sigma^2L}}\right)$ = 0

$$G_1(Q,R)=b_1esp(b_2L)\int_O^Q esp\left(\frac{\sigma^2b_2L}{2D^2}-\frac{b_2(R+Y)}{D}\right)$$

Thus

$$F\left(\frac{R+Y-\frac{\sigma b_2L}{D}-DL}{\sqrt{\sigma^2L}}\right)dy \qquad (5)$$

Define G_2 (R+y, L) such that

G_1(Q, R) = $\int_0^Q G_2(R+y,L)\,dy$ (6)

Thus

$$G_2(R+y,L)=\frac{D}{\sqrt{\sigma^2L}}\int_0^L\frac{C_B(L-z)}{\sqrt{2\pi}}\exp\frac{-1}{2}\left(\frac{R+y-Dz}{\sqrt{\sigma^2L}}\right)^2dz \qquad (7a)$$

Applying (5)

$$G_2(R+y,\, L) = b_1 e^{b_2 L} \exp\left(\frac{\sigma^2 b_2^2 L}{2D^2} - \frac{b_2(R+y)}{D} \right) F\left(\frac{R + y - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2}\, L} \right) \qquad (7b)$$

Let $V = \dfrac{R + y - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}}$

Substituting into (5) then

$$G_1(Q,R) = \sqrt{\sigma^2}\; L b_1\, esp(b_2 L) \int_{(R - \frac{\sigma^2 b_2 L}{D} - DL)/\sqrt{\sigma^2 L}}^{(R+Q - \frac{\sigma^2 b_2 L}{D} - DL)/\sqrt{\sigma^2 L}} \exp \frac{\sigma^2 b_2 L}{2D^2} \exp - \frac{b_2}{D} (\sqrt{\sigma^2}\, Lv + \frac{\sigma^2 L b_2}{D} + DL) F(v) dv$$

Simplifying we have

$$G_1(Q,R) = b_1 \sqrt{\sigma^2 L} * \exp\left(\frac{-\sigma^2 b_2 L}{2D^2} \right) * \int_{(R - \frac{\sigma^2 b_2 L}{D} - DL)/\sqrt{\sigma^2 L}}^{(R+Q - \frac{\sigma^2 b_2 L}{D} - DL)/\sqrt{\sigma^2 L}} \exp\left(\frac{-b_2 \sqrt{\sigma^2}\, Lv}{D}\right. F(v) dv \qquad (8)$$

Integrating by parts we have

$$G_1(Q,R)$$

$$= b_1 \sqrt{\sigma^2 L}\; \exp \frac{-\sigma^2 b_2 L}{2D^2} \left[\frac{-D}{b^2} \sqrt{\sigma^2 L}\; \exp \frac{-b_2 \sqrt{\sigma^2}\; Lv}{D} F(v) \right]_{\frac{R - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}}}^{(R+Q - \frac{\sigma^2 b_2 L}{D} - DL)/\sqrt{\sigma^2 L}}$$

$$\frac{-Db_1}{b_2} \frac{1}{\sqrt{2\pi}} \exp\; \frac{-\sigma^2 b_2 L}{2D^2} \int_{\frac{R - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-}{\sqrt{\sigma^2 L}} \frac{\sigma^2 b_2 L}{D}} \exp[\frac{-b_2 v \sqrt{\sigma^2} L}{D} - \frac{v^2}{2}] dv$$

Nothing that $\dfrac{-\sigma^2 b_2^2 L}{2D^2} - \dfrac{b_2^2 v \sqrt{\sigma^2 L}}{D} - \dfrac{v^2}{2}$

$$= -\frac{1}{2}\left(v^2 + \frac{2 b_2 v \sqrt{\sigma^2 L}}{D} + \frac{\sigma^2 L b_2^2}{D^2} \right)$$

$$= -\frac{1}{2}\left(v + b_2 \frac{\sqrt{\sigma^2 L}}{D} \right)^2$$

then

$$G_1(Q,R) = -\frac{Db_1}{b_2}\left[\exp\left[\frac{-\sigma^2 b_2^2 L}{2D^2} + \frac{b_2\sqrt{\sigma^2}\ L.v}{D}\right]F(v)\right]_{(R-\frac{b_2\sigma_2 L}{D}-DL)/\sqrt{\sigma^2}L}^{(R+q-\frac{b_2\sigma^2 L}{D}-DL)/\sqrt{\sigma^2 L}}$$

$$\frac{-Db_1}{b_2\sqrt{2\pi}}\int_{(R-\frac{\sigma^2 b_2 L}{D}-DL)/\sqrt{\sigma^2}L}^{(R+Q-\frac{\sigma^2 b_2 L}{D}-DL)/\sqrt{\sigma^2 L}} \exp\frac{-1}{2}\left(v + \frac{b_2\sqrt{\sigma^2 L}}{D}\right)dv$$

Let X = $v + \frac{b_2\sqrt{\sigma^2 L}}{D}$ and substituting into integral

Then

$$G_1(Q,R) = -\frac{Db_1}{b_2}\left[\exp\left(\frac{\sigma^2 b_2 L}{2D^2} + \frac{b_2\sqrt{\sigma^2 L}}{D}\right)F(v)\right]_{/\sqrt{\sigma^2 L}\ L\ \frac{\left(R-\frac{\sigma^2 b_2 L}{D}-DL\right.}{\sqrt{\sigma^2 L}}}^{\left(R+Q-\frac{\sigma^2 b_2 L}{D}-DL\right)} - \frac{Db_1}{b_2\sqrt{2\pi}}\int_{\frac{R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R-Q+DL}{\sqrt{\sigma^2 L}}} \exp-\left(\frac{x^2}{2}\right)dx$$

Integrating we have

$$G_1(Q,R) = \frac{-Db_1}{b_2}\left[\exp\frac{-\sigma^2 b_2 L}{2D^2}\exp\frac{b_2 v\sqrt{\sigma^2}L}{D}F(v)\right]_{\frac{(R-\frac{\sigma^2 L b_2}{D}-DL)}{\sqrt{\sigma^2 L}}}^{(R+Q-\frac{\sigma^2 b_2 L}{\sqrt{\sigma^2 L}}-DL)/\sqrt{\sigma^2 L}} + \frac{Db_1}{b_4}\left[F(v)\right]_{\frac{(R-DL}{\sqrt{\sigma^2 L}}}^{\frac{R+Q-DL}{\sqrt{\sigma^2 L}}} \quad (9)$$ Expanding

$$G_1(Q,R) = \frac{Db_1}{b_2}\left[\exp-\left(\frac{\sigma^2 L b_2^2}{2D^2}\right)\exp\frac{b_2}{D}\left(R - \frac{\sigma^2 L b_2}{D} - DL\right) * \left(\frac{R - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left. - \exp-\left(\frac{\sigma^2 b_2 L}{2D^2} + \frac{b_2}{D}\left(R + Q - \frac{\sigma^2 L b_2}{D} - DL\right)\right)F\left(\frac{R + Q - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2}L}\right)\right]$$

$$-\frac{Db_1}{b_1}\left(F\left(\frac{R-DL}{\sqrt{\sigma^2}L}\right) - F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \quad (10)$$

we are deriving the limit of $G_2(Q,R),\ b_1 e^{b_2 t} \to b_1 \ \ as\ \ b_2 \to 0$

noting from equation (8)

$$G_1(Q,R) = b_1\sqrt{\sigma^2}L\ \ esp\frac{(-\sigma^2 b_2 L)}{2D^2}\int_{(R-\frac{\sigma^2 b_2 L}{D}-DL)/\sqrt{\sigma^2 L}}^{(R+Q-\frac{\sigma^2 b_2 L}{D}-DL)/\sqrt{\sigma^2 L}} \exp\frac{-b_2\sqrt{\sigma^2 Lv}}{D}F(v)dv$$

$$\text{Lim}_{b_2 \to 0}\ G_1(Q,R) = \sqrt{\sigma^2 L}\quad b_1\int_{(R-DL)/\sqrt{\sigma^2 L}}^{(R+Q-DL)/\sqrt{\sigma^2 L}} F(v)dv \quad (11)$$

Noting that

$$\int_k^\infty F(v)\,dv = g(k) - F(k)$$

Hence $\lim_{b_2 \to 0} G_1(Q,R) = \sqrt{\sigma^2 L b_1}\left(g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)\right.$

$$-\left(\sqrt{\sigma^2 L}\; b_1 \quad \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad (12)$$

Let $\alpha(v) = \sigma^2 L(g(v) - v\;\; F(v))$ (13)

Thus

$$\lim_{b_2 \to 0} G_1(Q,R) = b_1\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \qquad (14)$$

Which can also be derived directly from (10)

From equation (10)

Lim

$$\underset{b_2 \to 0}{G_1}(Q,R) = \left[D\, b_1\left[-\left(\frac{\sigma^2 b_2 L}{D^2} + \frac{R}{D} - \frac{2\sigma^2 b_2 L}{D} - L\right) * \exp\left(\frac{-\sigma^2 b_2 L}{2D^2} - \frac{b_2}{D}\left(R - \frac{\sigma^2 b_2 L}{D} - DL\right)\right)\right.\right.$$

$$\left. F\left(\frac{R - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}}\right) + \frac{\sqrt{\sigma^2 L}}{D} esp\left(-\frac{\sigma^2 b_2 L}{2D^2} - \frac{b_2}{D}\left(R - \frac{\sigma^2 b_2 L}{D} - DL\right)\right) g\left(\frac{R - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}}\right)\right]_{R+Q}^{R} \qquad (15)$$

Simplifying

$$\underset{b_2 \to 0}{G_1}(Q,R) = Db_1\left[\left(\frac{-R}{D} F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + \frac{\sqrt{\sigma^2 L}}{D} g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) + \left(\frac{R+Q}{D}\right) F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right.\right.$$

$$\left.\frac{\sqrt{\sigma^2 L}}{D} g\left(\frac{R+Q-DL}{\sigma^2 L}\right)\right] \qquad (16)$$

Simplifying

$$\lim_{b_2 \to 0} G_1(Q,R) = \sqrt{\sigma^2} L b_1\left(g\left(\frac{R-DL}{\sigma^2 L}\right) - \left(\frac{R-DL}{D}\right) F\left(\frac{R-DL}{\sigma^2 L}\right)\right) \qquad (17)$$

$$-\sqrt{\sigma^2 L} b_1\left(g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) - \left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) F\left(\frac{R+Q-DL}{\sigma^2 L}\right)\right. \qquad (18)$$

As before letting $\alpha(V) = \sqrt{\sigma^2 L}(g(V) - V(F(V\;\;)))$

Thus

$$\lim_{b_2 \to 0} G_1(Q,R) = b_1\left(\alpha\left(\frac{R-DL}{\sigma^2 L}\right) - \alpha\left(\frac{R+Q-DL}{\sigma^2 L}\right)\right) \qquad (19)$$

Which agrees with equation (15)

Number of cycles per year = $\frac{D}{Q}$

Hence expected cost of backorders per year

$$\frac{D}{Q} G_1(Q,R)$$

Let B(Q,R) be the expected number of backorders at any time. (Ref 1. Hadley and Whitin)

Let $G_3(x)$ the probability density function of the quantity on hand x at anytime t. x could be less than the re-order level, R or above the reoder level R.

$G_3(x)$ = probability that the system was in state v. probability that v-x was demanded and average over the states of v.

If x lies below re-order level R, $G_1(x) = \frac{1}{Q}\int_R^{R+Q} esp - \tfrac{1}{2}\left(\sqrt{\frac{v-x-DL}{\tau^2 L}}\right)^2 dv \qquad 0 < x < R$

$$Let\, Z = \frac{v - xDL}{\sqrt{\sigma^2 L}}\, and\, k = \frac{R - DL}{\sqrt{\sigma^2 L}} \qquad (20)$$

$$then \quad G_1(x) = \sqrt{2\pi}\, \frac{1}{Q}\int_{k-\frac{x}{\sqrt{\tau^2 L}}}^{k+\frac{Q-x}{\sqrt{\tau^2 L}}} \exp - \frac{z^2}{2} dz \qquad (21)$$

Integrating

$$G_3(x) = \frac{1}{Q}\left(f\left(k - \frac{x}{\sqrt{\tau^2 L}} \right) - f\left(k + \frac{Q-x}{\sqrt{\sigma^2 L}} \right)\right) \qquad 0 < x < R \qquad (22)$$

If x lies above R

$$G_3(x) = \sqrt{2\pi\sigma^2 L}\,\frac{1}{Q}\int_x^{R+Q} \exp - \frac{1}{2}\left(\frac{v - xDL}{\sqrt{\sigma^2 L}}\right) dv \qquad R < x < R + Q \qquad (23)$$

Expressing in standard deviations of stock

$$G_3(x) = \sqrt{2\pi}\,\frac{1}{Q}\int_{-\frac{DL}{\sqrt{\sigma^2 L}}}^{K+\frac{Q-x}{\pi^2 L}} \exp - \frac{z^2}{2} dz \qquad R < x < R + Q$$

$$= \frac{1}{Q}\left(f\left(-\frac{DL}{\sqrt{\sigma^2 L}} \right) - f\left(k + \frac{Q-x}{\sqrt{\sigma^2 L}} \right)\right) \qquad (24)$$

We shall assume that $f\left(-\frac{DL}{\sqrt{\sigma^2 L}} \right) = 1$

Hence

$$G_3(x) = \frac{1}{Q}\left(F\left(k - \frac{x}{\sqrt{\sigma^2 L}} \right) - F\left(k + \frac{(Q-x)}{\sqrt{\sigma^2 L}} \right)\right) \qquad 0 < x < R$$

$$G_3(x) = \frac{1}{Q}\left(1 - F\left(k + \frac{Q-x}{\sigma^2 L} \right)\right) \qquad R < x\ R + Q \qquad (25)$$

Let y be the number of backorder at time t given that the inventory position was v at time 0. Obviously y + v must have been demanded.

Let G_4 (y) be the density function that y is the number of backorders on hand at any time t.

G_4 (y) averaging over all the states of v.

Hence

$$G_4(y) = \frac{1}{Q\sqrt{2\pi\sigma^2 L}} \int_R^{R+Q} esp - \tfrac{1}{2}\left(\frac{v + y - DL}{\sqrt{\sigma^2 L}}\right)^2 dv \qquad y > 0 \qquad (26)$$

Expressing in standard deviations of stock

$$G_4(y) = \frac{1}{Q\sqrt{2\pi}} \int_{\frac{k-y}{\sqrt{\sigma^2 L}}}^{\frac{k+Q-y}{\sqrt{\sigma^2 L}}} \exp - \frac{Z^2}{2} \quad dZ$$

Hence

$$G_4(y) = \frac{1}{Q}\left(F\left(k + \frac{y}{\sqrt{\sigma^2 L}} \right) - F\left(k + \left(\frac{y+Q}{\sqrt{\sigma^2 L}} \right)\right)\right) \qquad (27)$$

Y is the backorders at anytime and $G_2(y)$ is the probability that the backorders at any time is y.
Hence

$$B(Q,K) = \int_0^\infty yG_2(y)\, dy$$

Substituting for $G_2(y)$ in the above expression

$$B(Q,k) = \frac{1}{Q}\int_0^\infty yF\left(k + \frac{y}{\sqrt{\tau^2 L}} \right) dy - \frac{1}{Q}\int_0^\infty yF\left(k + \frac{Q+y}{\sqrt{\sigma^2 L}} \right) dy \qquad (28)$$

Substituting Z for $k + \frac{y}{\sqrt{\sigma^2 L}}$ *and* Z *for* $k + \frac{Q+y}{\sqrt{\sigma^2 L}}$

Respectively and expanding the expression

$$B(Q,k) = \frac{\sigma^2 L}{Q}\int_k^\infty zf(z)\, dz - \frac{k\sigma^2 L}{Q}\int_k^\infty f(z) dz$$

$$- \frac{\sigma^2 L}{Q}\int_{k+\frac{Q}{\sqrt{Q^2 L}}}^\infty Zf(z)\, dZ + \frac{\sigma^2 L}{Q}\int_{k+\frac{Q}{\sqrt{\tau^2 L}}}^\infty \left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right) F(z)\, dz \qquad (29)$$

Noting that

$$\int_k^\infty x^n F(x)\, dx = \left[\frac{x^{n+1}}{n+1} F(x) \right]_k^\infty + \int_k^\infty \frac{x^{n+1}}{n+1}\, g(x)\, dx$$

Then when n=1

$$\int_k^\infty z\, F(z)\, dz = \tfrac{1}{2}\left(1 - k^2\right) F(k) + kg(k))$$

And when n=0

$$\int_k^\infty F(z)\, dz = g(k) - kf(k) \qquad (30)$$

Substituting (30) into (29)

$$B(Q,K) = \frac{\sigma^2 L}{2Q}\left[(1-k^2) F(k) + kg(k) \right] - \frac{k\sigma^2 L}{Q}\left(g(k) - kF(k) \right)$$

$$- \frac{\sigma^2 L}{2Q}\left[\left(1 - \left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right)^2 \right) F\left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right) + \left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right) g\left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right) \right]$$

$$+ \left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right) \frac{\sigma^2 L}{Q}\left[g + \frac{Q}{\sqrt{\sigma^2 L}} \right) - \left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right) F\left(k + \frac{Q}{\sigma^2 L} \right) \right]$$

$$the\ \beta(V) = \frac{\sigma^2 L}{2}\left[\left(1 + V^2 \right) F(v) - Vg(v) \right] \qquad (31)$$

Hence

$$B(Q,k) = \frac{1}{Q}\left(\beta(k) - \beta\left(k + \frac{Q}{\sqrt{\sigma^2 L}} \right) \right) \qquad (32)$$

Then the inventory costs for model (Q,R) with exponential cost terms and noting that $k=(R-DL)/\sqrt{\sigma^2 L}$

$$C=\frac{DS}{Q}+\frac{Qhc}{2}+hc\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)+hcB(Q,R)+\frac{D}{Q}G_1(Q,R)+\frac{Ds}{Q}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-\alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \quad (33)$$

First order derivatives

In order to be able to obtain the first derivatives of C, the first order derivatives of G_1 (Q,R) with respective to Q and R would be required.

Differentiate G_1(Q,R) with respect to Q and applying equation (5)

$$\frac{\partial G_1(Q,R)}{\partial Q}=b_1\exp b_2\ L\exp\left(\frac{\sigma^2 b_2^2 L}{2D^2}-b_2\frac{(R+Q)}{D}\right)F\left(\frac{R+Q-\frac{\sigma^2 b_2 L}{D}-DL}{\sqrt{\sigma^2 L}}\right) \quad (34)$$

Differentiating G_1 (Q,R) with respect to R and from (9)

$$\frac{\partial G_1(Q,R)}{\partial R}=\frac{Db_1}{b_2}\left[\frac{b_2}{D}\exp-\left(\frac{\sigma^2 b_2 L}{2D^2}+\frac{b_2}{D}\left(R-\frac{\sigma^2 b_2 L}{D}-DL\right)\right)F\left(\frac{R-\frac{\sigma^2 b_2 L}{D}-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left.-\frac{1}{\sqrt{2\pi\sigma^2 L}}\exp\left(\frac{\sigma^2 b_2 L}{2D^2}+\frac{b_2}{D}\left(R-\frac{\sigma^2 b_2}{D}-DL\right)\right)esp-\frac{1}{2\sigma^2 L}\left(R-\frac{\sigma^2 b_2 L}{D}-DL\right)^2\right]$$

$$-\frac{-Db_1}{b_2}\left[\frac{b_2}{D}\exp-\left(\frac{\sigma^2 b_2 L}{2D^2}+\frac{b_2}{D}\left(R+Q\frac{-\sigma^2 b_2 L}{D}-DL\right)\right)*F\left(\frac{R+Q-\frac{\sigma^2 b_2 L}{D}-DL}{\sqrt{\sigma^2 L}}\right)\right.$$

$$\left.-\frac{1}{\sqrt{2\pi\sigma^2 L}}esp-\left(\frac{\sigma^2 b_2 L}{2D^2}+\frac{b_2}{D}\left(R+Q-\frac{\sigma^2 b_2 L}{D}-DL\right)\right)\exp-\frac{1}{2\sigma^2 L}\left(R+Q-\frac{\sigma^2 b_2 L}{D}-DL\right)^2\right]$$

$$+\frac{Db_1}{b_2\sqrt{\sigma^2 L}}\left(g\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)-g\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right) \quad (35)$$

Simplifying

$$\frac{\partial G_1(Q,R)}{\partial R} = \Bigg[\; b_1 \exp\left(\frac{\sigma^2 b_2 L}{2D^2} - \frac{b_2}{D}(R-DL)\right) F\left(\frac{R - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}}\right)$$

$$- \frac{Db_1}{b_2\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R-DL}{\sigma^2 L}\right)^2 \; \Bigg]$$

$$-\Bigg[\; b_1 \exp\left(\frac{\sigma^2 b_2 L}{2D^2} - \left(\frac{R+Q-DL}{D}\right) b_2 \; \right) F\left(\frac{R+Q - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2} L}\right) - \frac{Db_1}{b_2\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2 \; \Bigg]$$

$$+\left(\frac{Db_1}{b_2\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R-DL}{\sigma^2 L}\right)^2 - \frac{Db_1}{b_2\sqrt{2\pi\sigma^2 L}} \exp - \frac{1}{2}\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)^2 \; \right)$$

Simplifying

$$\frac{\partial G_1(Q,R)}{\partial R} = b_1 \exp \frac{\sigma^2 b_2 L}{2D^2}\Bigg[\exp - \frac{b_2}{D}(R-DL) F\left(R - \frac{\frac{\sigma^2 b_2 L}{D}}{D} - DL) - \exp - \frac{b_2}{D}(R+Q-DL) \right.$$

$$\left. F(\frac{R+Q - \frac{\sigma^2 b_2 L}{D} - DL}{\sqrt{\sigma^2 L}} \quad \right) \Bigg] \tag{36}$$

Noting that

$$\frac{\partial \alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)}{\partial R} = -F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right)$$

$$and \quad \frac{\partial \alpha\left(\frac{R+Q-DL}{\sigma^2 L}\right)}{\partial R} = -F\left(\frac{R+Q-DL}{\sigma^2 L}\right)$$

Then the first order derivatives of C are

$$\frac{\partial C}{\partial Q} = -\frac{DS}{Q^2} + \frac{hc}{2} - P\frac{G_1(Q,R)}{Q^2} + \frac{DdG_1(Q,R)}{Q\partial Q} - \frac{-DS}{Q}\left(\alpha\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - \alpha\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)\right)$$

$$+ \frac{Ds}{Q} F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right)$$

$$\frac{\partial C}{\partial R} = hc + \frac{D\partial G_2(Q,R)}{Q\partial R} - \frac{Ds}{Q}\left(F\left(\frac{R-DL}{\sqrt{\sigma^2 L}}\right) - F\left(\frac{R+Q-DL}{\sqrt{\sigma^2 L}}\right) \right) \tag{37}$$

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