Vibration of three-layered cylindrical shell with functionally graded middle layer for various volume fraction laws

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Abstract:

Vibration of cylindrical shells is accomplished for their involvement in various areas of engineering and technology. Shell vibration behavior depends upon on geometrical material parameters. Materials in functionally graded forms are more progressive ones. They provide the maximum stability of a physical system. There is graduation distribution of constituent materials in functionally graded materials and is controlled by polynomial, exponential and trigonometric volume exponent fraction laws. In the present study a cylindrical shell is composed of three layers whereas the middle layer consists of functionally graded material and the extreme layer are of isotropic nature. Material composition of the FG layer is governed by polynomial, exponential and trigonometric volume fraction exponent laws. Impact of these laws is examined on shell vibration frequencies for different physical parameters. Love's thin shell theory is adopted for shell motion equations. The Rayleigh-Ritz technique is applied to form the shell frequency equation which is solved by MATLAB software. The validity and accuracy of this method is investigated for a number of comparisons of numerical results.

1.1 Introduction

Cylindrical shells are essential components in the field of technology as well as that of engineering. Vibrations of cylindrical shells have been extensively studied for their simple geometrical designing. So a huge amount of research on them is seen in open literature. Egle *et al.* **[1]** examined free vibrations of orthogonally inflexible cylindrical shells where rigidness has been treated as distinct elements**.** Sharma *et al.* **[2]** investigated vibrations of cylindrical shells for clamped-free boundary conditions by using Rayleigh-Ritz technique**.** The vibration of cylindrical shells with intermediary supports was examined by Swaddiwudhpong *et al.* **[3].** Vibrations of functionally graded cylindrical shells were investigated by Loy *et al.***[4]** and Pardhan *et al.* **[5]** for various physical parameters and several boundary conditions**.** Li *et al*. **[6]** examined vibrations of circular cylindrical shells with functionally graded materials middle layer for simply supported end conditions The idea of tri-layered cylindrical shells with intermediate layer of FGM was given by Li and Batra **[7]** for studying axial buckling of cylindrical shells**.** They also used Love's approximation for strain and curvature-displacement relationships for shells The idea of tri layered cylindrical shells with intermediate layer of FGM was given by Batra **[7]** for studying axial buckling of cylindrical shells and they investigated this aspect of dynamical study of the shells. Bing *et al.* **[8]** examined vibration frequencies of thin walled cylindrical shells for different edge condition**.** Shao and Ma **[9]** investigate the vibration analysis of those cylindrical shells split into thin layer and used Fourier series expression method for SS-SS, C-C, C-F and C-SS boundary conditions**.** Naeem *et al.* **[10]** employed the Ritz formulation to investigate vibration of natural frequency characteristic of FG cylindrical shells. Naeem *et al.* **[11]** established the equation of FGM shells in eigenvalue expression to observe their frequencies.

In this study vibration characteristics of three layered cylindrical with functionally graded middle layer are investigated. The frequencies analysis of two layers cylindrical shells was examined by Arshad *et al*. **[12]** in which one layer was FG layer and other layer was of homogeneous materials. Iqbal *et al*. **[13]** examined vibrations of FG cylindrical shells applying the wave propagation technique. The generalized differential quadrature method was applied to examine the vibration characteristics of functionally graded materials cylindrical shells by Naeem *et al*. **[14]**. Sofiyev *et al*. **[15]** examined the constancy of FG cylindrical shells attached to combine loads with various ends conditions and resting on elastic foundations. Vel **[16]** employed the elasticity solution technique to observe free and forced vibration of cylindrical shells. These shells were estimated by SS-SS boundary condition. Shah *et al*. **[17]** applied exponential volume fraction law to observe the cylindrical shell's vibration with FGM. Warburton *et al*. **[18]** investigated the appearance of frequency variations with the circuit wave and expressed the frequency in the form of shell energies. Vibration of spinning cylindrical shells was examined by Mehparvar **[19].** The shells were constructed from FGM. They used the higher ordered theory for shell inflections with the use of energy Hamilton's principle to obtain the shell dynamical equations. Material grade and turning velocity effect on the frequencies of shells. The vibration of cylindrical shells which are containing FGM was observed by Lam *et al.* **[20].** Their purpose was to check the effect of FGM on vibration characteristics of the shells. Their composition was maintained by volume fraction power law of distribution of materials in the radial direction. Yamanouchi *et al.***[21]** and Koizumi **[22]** studied the structure and design of FGMs.

In this paper vibration of three layered cylindrical shells are analyzed for various shell parameters. The shell thickness consists of three layers where materials of the extremes are of isotropic. The middle layer consists of functionally graded materials. The shell problem has been written in the integral form by considering expressions of kinetic and strain energies for a cylindrical shell. The shell frequency equation is formed by applying the Raleigh-Ritz technique. The estimation of axial modal dependence is done by characteristic beam functions. These functions satisfy boundary conditions. Results are obtained for simply supported- simply supported, clamped-clamped, clamped-free and clamped-simply supported boundary conditions. Comparisons of results determined by this procedure are done with those ones found in literature to verify the validity and efficiency of this technique and accuracy of the results.

1.2 Theoretical formulation:

Figure 1 represents the geometry of a cylindrical shell. $L, h, R,$ stand for its geometrical quantities viz.; length, thickness and mean radius respectively while E, v and ρ designate explained Young's modulus, the Poisson ratio and the mass density respectively. The triplet (x, θ, z) defines an orthogonal coordinate system and they lie at the mid plane of the cylindrical shell. They describe the coordinates in the longitudinal, tangential and transverse directions respectively. The functions $u(x,\theta,z)$, $v(x, \theta,z)$ and $w(x, \theta, z)$ indicate for the longitudinal, tangential and transverse displacements from the mid surface of the shell.

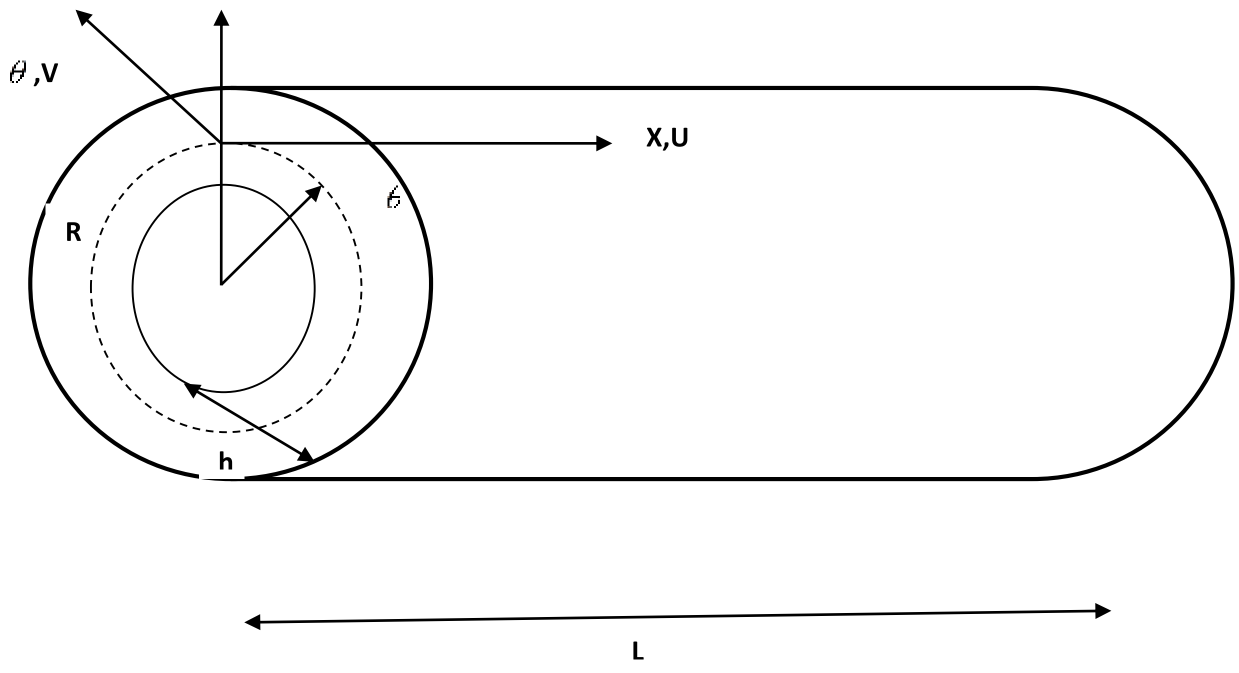

Figure: 1 Coordinate system and shell geometry

For a vibrating thin cylindrical shell, its strain energy, expressed by U is stated as:

$$U = \frac{1}{2}\int_0^L\int_0^{2\pi}\left\{A_{11}e_1^2 + A_{22}e_2^2 + 2A_{12}e_1e_2 + A_{66}\gamma^2 + 2B_{11}e_1k_1 + 2B_{12}e_1k_2\right.$$

$$\left.+2B_{12}e_2k_1 + 2B_{22}e_2k_2 + 4B_{66}\gamma\tau + D_{11}k_1^2 + D_{22}k_2^2 + 2D_{12}k_1k_2 + 4D_{66}\tau^2\right\}Rd\theta dx \quad (1)$$

where the stress where e_1, e_2 and γ define the reference surface strains, k_1, k_2 and τ represent the surface curvatures and where A_{ij}, B_{ij} and D_{ij} $(i, j = 1,2$ and $6)$ are associated with the extensional , coupling and bending stiffness respectively and are stated as:

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ij}dz, B_{ij} = \int_{-h/2}^{h/2} Q_{ij}zdz, D_{ij} = \int_{-h/2}^{h/2} Q_{ij}z^2dz, \quad (2)$$

The reduced material stiffness $Q_{ij}(i, j = 1,2$ and $6)$ for isotropic materials are described as:

$$Q_{11} = Q_{22} = \frac{E}{1-v^2}, Q_{12} = \frac{vE}{1-v^2}, Q_{66} = \frac{E}{2(1+v)} \quad (3)$$

for isotropic cylindrical shells the coupling stiffness considered equal zero and for the shells formed by FGM they considered non-zero. For the cylindrical shells which fabricated by functionally graded materials their values depend on the position FGM. The negativity and positivity of coupling stiffness exist due to the irregularity of characteristics of materials at mid plane when reduced stiffness produced by physical properties of functionally graded materials.

Also the kinetic energy of the cylindrical shell, denoted by T, is written as

$$T=\frac{1}{2}\int_0^L\int_0^{2\pi}\rho_t\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]Rd\theta dx \tag{4}$$

where t denotes the time variable and ρ_t represents the mass density per unit length and is written as

$$\rho_t=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho dz \tag{5}$$

where ρ stand for the mass density.

1.3 Love's shell theory

Several shell theories have been found in the open literature. Kirchhoff's assumption is the basis for all shell theories. This assumption states that "Normal to the original mid-surface of a shell retains its normal position, suffer no change in length during deformation". Shell theory due Love is the pioneering one and all other modern theories have designed from it by modifying some physical terms. The formulas for strain and curvature–displacements are adopted from Love's shell theory to solve the present shell problem and are written as:

$$\{e_1,e_2,\gamma\}=\left\{\frac{\partial u}{\partial x},\frac{1}{R}\left(\frac{\partial v}{\partial\theta}+w\right),\left(\frac{\partial v}{\partial x}+\frac{1}{R}\frac{\partial u}{\partial\theta}\right)\right\} \tag{6}$$

$$\{k_1,k_2,\tau\}=\left\{-\frac{\partial^2 w}{\partial x^2},-\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right),-\frac{1}{R}\left(\frac{\partial^2 w}{\partial x\partial\theta}-\frac{\partial v}{\partial x}\right)\right\} \tag{7}$$

These expressions for the surface strains e_1, e_2, and γ and the curvatures k_1, $k_{2,}$ and τ from the relations (6) and (7) respectively are replaced into Equ.(1), the expression for strain energy, U attains the following form:

$$U=\frac{1}{2}\int_0^L\int_0^{2\pi}\{A_{11}\left(\frac{\partial u}{\partial x}\right)^2+\frac{A_{22}}{R^2}\left(\frac{\partial v}{\partial\theta}+w\right)^2+\frac{2A_{12}}{R}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial\theta}+w\right)$$

$$+A_{66}\left(\frac{\partial v}{\partial x}+\frac{1}{R}\frac{\partial u}{\partial\theta}\right)^2-2B_{11}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)-\frac{2B_{12}}{R^2}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial x}\right)$$

$$-\frac{2B_{12}}{R}\left(\frac{\partial v}{\partial\theta}+w\right)\left(\frac{\partial^2 w}{\partial x^2}\right)-\frac{2B_{22}}{R^3}\left(\frac{\partial v}{\partial\theta}+w\right)\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right)-\frac{8B_{66}}{R}$$

$$\times\left(\frac{\partial v}{\partial x}+\frac{1}{R}\frac{\partial u}{\partial\theta}\right)\left(\frac{\partial^2 w}{\partial x\partial\theta}-\frac{\partial v}{\partial x}\right)+D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{D_{22}}{R^4}\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right)^2$$

$$+\frac{2D_{12}}{R^2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial v}{\partial\theta}\right)+\frac{4D_{66}}{R^2}\left(\frac{\partial^2 w}{\partial x\partial\theta}-\frac{\partial v}{\partial x}\right)^2\}Rd\theta dx \tag{8}$$

The Lagrange energy functional, symbolized by $\prod$ for a cylindrical shell is described by the difference of its strain and kinetic energies as:

$$\prod = T - U \tag{9}$$

The Raleigh-Ritz technique is used to examine the vibration of cylindrical shells. The deformation of cylindrical shells in longitudinal, tangential and transverse direction describe in the form of shell motion's equations with particular variables. Many kinds of mathematical functions are used to measure the axial modal dependence. The boundaries conditions of cylindrical shells are satisfied by them.

1.4 Modal displacement functions

The unidentified displacement functions $u(x,\theta, z)$, $v(x, \theta, z)$, $w(x, \theta, z)$, showing deformations in the longitudinal, tangential and transverse directions are supposed in such shapes that the separation of the special and temporal variables is performed. This process is done by classical technique of separation of variables used for solving partial differential equations. The substitution of the presumed shapes of the modal displacement functions are made into the shell governing equations and a system of simultaneous equations is obtained in the vibration amplitude coefficient by the Rayleigh-Ritz method. The axial modal dependence related to the unknown functions is determined those functions which meet boundary conditions described for a cylindrical shells. The following models for the modal deformation function are mentioned for axial, tangential and temporal variables:

$$u(x,\theta,z) = AU(x)\sin n\theta \sin \omega t \tag{10a}$$

$$u(x,\theta,z) = AU(x)\sin n\theta \sin \omega t \tag{10b}$$

$$w(x,\theta,z) = CW(x)\sin n\theta \sin \omega t \tag{10c}$$

where ω denotes the frequency of the cylindrical shell and n is the circumferential wave number. The coefficients A, B, C show the vibration amplitudes in the longitudinal, tangential and transverse directions respectively.

Substituting the above expressions of the shell energies into Equation (9), the new expression for the Lagrange functional is achieved as:

$$\prod = \frac{R\rho_t}{2}\int_0^L \left\{U^2(x)A + V^2(x)B + W^2(x)C\right\}$$

$$-\frac{R\pi}{2}\int_0^L A_{11}A\left(\frac{dU}{dx}\right)^2 + \frac{A_{22}}{R^2}\left(-nBV(x) + CW(x)\right)^2 + A_{66}\left(B\frac{dV}{dx} + \frac{n}{R}AU(x)\right)^2$$

$$+A_{66}\left(B\frac{dV}{dx} + \frac{n}{R}AU(x)\right)^2 + \frac{2A_{12}A}{R}\left(-nBV(x)\frac{dU}{dx} + CW(x)\frac{dU}{dx}\right)$$

$$+2B_{11}\left(-AC\frac{dU}{dx}\frac{d^2W}{dx^2}\right) + \frac{2A_{12}A}{R^2}\left(-ACn^2\frac{dU}{dx} + nABV(x)\frac{dU}{dx}\right) + 2B_{12}\left(nBCV(x)\frac{d^2W}{dx^2} - C^2W(x)\frac{d^2W}{dx^2}\right)$$

$$-\frac{2B_{22}}{R^3}\left(-n^2C^2W^2(x) + n^3BCW(x)V(x) + nBCW(x)V(x) - n^2B^2V^2(x)\right)$$

$$-\frac{4B_{66}}{R}\left(nBC\frac{dV}{dx}\frac{dW}{dx} - B^2\left(\frac{dV}{dx}\right)^2 + \frac{n^2}{R}ACU(x)\frac{dW}{dx} - \frac{n}{R}ABU(x)\frac{dV}{dx}\right)$$

$$+D_{11}C^2\left(\frac{d^2W}{dx^2}\right)^2 + 2D_{12}\left(-n^2C^2W(x)\frac{d^2W}{dx^2} + nBCV(x)\frac{d^2W}{dx^2}\right) - \frac{4}{R}D_{66}$$

$$\left(n^2C^2\left(\frac{dW}{dx}\right)^2 + B^2\left(\frac{dv}{dx}\right)^2 - 2nBC\frac{dV}{dx}\frac{dW}{dx}\right)dx \quad (11)$$

Applying the Rayleigh- Ritz method, the process of minimization is applied to the Lagrange functional $\prod$ and is partially differentiated with regard to the vibration amplitude coefficients *A*, *B* and *C*. So doing process of extremization of $\prod$, the following required extrema conditions are obtained:

$$\frac{\partial\prod}{\partial A} = \frac{\partial\prod}{\partial B} = \frac{\partial\prod}{\partial C} = 0 \quad (12)$$

1.5 Derivation of the shell frequency equation

The point when terms of these compelling conditions adjusted in particular shape then shell recurrence mathematical statement is discovered. Three concurrent mathematical statements in *A*, *B*, *C* are acquired as:

$$C_{11}A + C_{12}B + C_{13}C = 0 \quad (13)$$

$$C_{21}A + C_{22}B + C_{23}C = 0 \quad (14)$$

$$C_{31}A + C_{32}B + C_{33}C = 0 \quad (15)$$

where the coefficients C_{ij}*'s* =(*i,j=1,2,3*) are listed in appendix. The above equations can be written in the matrix form as

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (16)$$

This represents the frequency equation in the eigenvalue problem form. The condition of making the determinant of the matrix coefficients zero is applied for non-trivial solution for achieving the frequency equation.

1.6 Polynomial volume fraction law

The properties of functionally graded materials vary for temperature and they are originating in the field of high thermal condition. If the material property is denoted by *P* which is function of the absolute temperature *T(K)*. Then Touloukian (1973) stated as:

$$P = P_0(P_{-1}T^{-1} + P_1T + P_2T^2 + P_3T^3) \quad (17)$$

where the thermal coefficients are indicated by P_0, P_{-1}, P_1, P_2 and P_3 while *T* indicates the temperature at absolute scale. The material properties of a functionally graded constituent material for a cylindrical shell are functions of both temperature and their volume fractions. The succeeding material of a functionally graded material is described as:

$$P = \sum_{j=1}^{k} P_j V_{f_j}, \quad (18)$$

where the materials characteristics are mentioned by $P_j\,'s$ and the volume fraction of FGM denoted by $V_{f_j}\,'s$, . Their sum always equal to one

i.e. $$\sum_{j=1}^{K} V_{f_j} = 1 \quad (19)$$

V_f denotes the volume fraction of a FG material. It can be written as:

$$V_f = \left[\frac{z-h_2}{h_3-h_2}\right]^N \quad (20)$$

The thickness of cylindrical shell denoted by *h* and power-law exponent by *N* and its value always lie between zero and infinity. FGM are composition of two materials. For a FG cylindrical shell E , ν & ρ are expressed as:

$$E = (E_1 - E_2)\left[\frac{Z-h_2}{h_3-h_2}\right]^N + E_2 \quad (21)$$

$$V = (V_1 - V_2)\left[\frac{Z-h_2}{h_3-h_2}\right]^N + V_2 \quad (22)$$

$$P = (P_1 - P_2)\left[\frac{Z-h_2}{h_3-h_2}\right]^N + P_2 \quad (23)$$

where $z= -h/3$, $E=E_2$, $v=v_2$denotes the materials used for $M_2$and $z= -h/3$, $E=E_1$, $v=v_1$ describe the materials for M_1. Both the materials present on the inward and outward surfaces of cylindrical shells can change their materials characteristics by interchanging themselves. The cylindrical shells with FGM are usually in-homogeneous shell. When the thickness of a shell toward its radius ratio is less than 0.05 then the theory of classical thin-walled cylindrical shell is applicable.

1.7 Exponential volume fraction law

Arshad *et al.* **[10]** modified the polynomial volume fraction law (20) and framed it in the exponential expression as:

$$V_j = 1 - e^{-\left(z/h + 0.5\right)^N}. \quad (24)$$

where $e = 2.718\ldots$ is the usual natural base. Further formula is amended and a more general base $(b > 0)$ is established and a new expression is written as:

$$V_j = 1 - b^{-\left(z/h + 0.5\right)N}, \quad (25)$$

Thus formulae for the effectual material properties: the effective Young's modulus E, the Poisson ratio v and the mass density ρ for a FG are written as:

$$E = \left(E_1 - E_2\right)\left(1 - b^{-\left(z/h+0.5\right)^N}\right) + E_2, \quad (26a)$$

$$v = \left(v_1 - v_2\right)\left(1 - b^{-\left(z/h+0.5\right)^N}\right) + v_2, \quad (26b)$$

$$\rho = \left(\rho_1 - \rho_2\right)\left(1 - b^{-\left(z/h+0.5\right)^N}\right) + \rho_2, \quad (26c)$$

when $z = -h/2$, $E = E_2$, $v = v_2$, and $\rho = \rho_2$, when $z = h/2$, $E = \left(E_1 - E_2\right)\left(1 - b^{-1}\right) + E_2$, $v = \left(v_1 - v_2\right)\left(1 - b^{-1}\right) + v_2$, and $\rho = \left(\rho_1 - \rho_2\right)\left(1 - b^{-1}\right) + \rho_2$,

The above relations express M_2 present at the inward surface while M_1 at outward surface of the cylindrical shells.

1.8 Trigonometric volume fraction law

This law obtained by making some changing in the formulae defines in (20) and (25) for cylindrical shell with FG layer related to M_1 and M_2 can be defined as:

$$V_{f1} = \sin^2\left[\left(z/h + 0.5\right)^N\right] \quad (27)$$

$$V_{f2} = \cos^2\left[\left(z/h + 0.5\right)^N\right] \quad (28)$$

The formulae (3.13) and (3.14) show that

$$V_{f1}+V_{f2}=1 \tag{29}$$

where N is a positive real number. The conclude materials for this law can also express like other two laws for cylindrical shells with FG

$$E=(E_1-E_2)\sin^2\left[\left(z/h+0.5\right)^N\right]+E_2 \tag{30a}$$

$$v=(v_1-v_2)\sin^2\left[\left(z/h+0.5\right)^N\right]+v_2 \tag{30b}$$

$$\rho=(\rho_1-\rho_2)\sin^2\left[\left(z/h+0.5\right)^N\right]+\rho_2 \tag{30c}$$

From formulae (3.16), when $z=-h/2,\ E=E_2,\ v=v_2,\ \rho=\rho_2$ and when $z=h/2,\ E=(E_1-E_2)\left[\sin^2(1)\right]+E_2$, $v=(v_1-v_2)\left[\sin^2(1)\right]+v_2,\ \rho=(\rho_1-\rho_2)\left[\sin^2(1)\right]+\rho_2$, Thus at $z=-h/2$, M_2 is attached at inward side but when $z=h/2$ then the characteristics material are obtained by both M_1 and M_2 materials present at outward surface of the cylindrical composed with functionally graded material.

1.9 Material Stiffness for three-layered cylindrical shells

The thickness layer of the cylindrical shell is divided into three layers. Thicknesses of interior, intermediate and exterior layers are h_1, h_2 and h_3 respectively. For simplicity, thickness of each layer is of the thickness h/3. According to this configuration, the coefficients of extensional, coupling and bending stiffness $A_{ij,}$, B_{ij} and D_{ij} are modified as

Here *E, E_2 and E_2* are Young's moduli, *N* is power-law exponent and v_f volume fractions while v,v_1 and v_2 are Poisson ratios.

$$A_{11}=A_{22}=\frac{E(h_2-h_1)}{1-v_1{}^2}+\frac{E(h_3-h_2)}{1-v_f{}^2}\left[\frac{(E_1-E_2)}{N+1}+E_2\right]+\frac{E(h_4-h_3)}{1-v_2{}^2}$$

$$A_{12}=\frac{vE(h_2-h_1)}{1-v_1{}^2}+\frac{E(h_3-h_2)v_f}{1-v_f{}^2}\left[\frac{(E_1-E_2)}{N+1}+E_2\right]+\frac{vE(h_4-h_3)}{1-v_2{}^2}$$

$$A_{66}=\frac{E(h_2-h_1)}{2(1+v_1)}+\frac{(h_3-h_2)}{2(1+v_f)}\left[\frac{(E_1-E_2)}{N+1}+E_2\right]+\frac{E(h_4-h_3)}{2(1+v_2)}$$

$$B_{11}=B_{22}=\frac{E(h^2{}_2-h^2{}_1)}{2(1-v^2{}_1)}+\frac{1}{1-v_f{}^2}\left[(E_1-E_2)\left\{\frac{(h^2{}_3-h^2{}_2)}{N+2}+\frac{h_2(h_3-h_2)}{N+1}\right\}+\frac{E(h^2{}_3-h^2{}_2)}{2}\right]+\frac{E(h^2{}_4-h^2{}_3)}{(1-v^2{}_2)}$$

$$B_{12}=\frac{vE\left(h_2{}^2-h_1{}^2\right)}{2\left(1-v_1{}^2\right)}+\frac{v_f}{1-v_f{}^2}\left[(E_1-E_2)\left\{\frac{(h_3-h_2)^2}{N+2}+\frac{h_2(h_3-h_2)}{N+1}\right\}+\frac{E_2\left(h_3{}^2-h_2{}^2\right)}{2}\right]+\frac{vE\left(h_4{}^2-h_3{}^2\right)}{2\left(1-v_2{}^2\right)}$$

$$B_{66}=\frac{E\left(h_2{}^2-h_1{}^2\right)}{4(1+v_1)}+\frac{v_f}{2(1+v_f)}\left[(E_1-E_2)\left\{\frac{(h_3-h_2)^2}{N+2}+\frac{h_2(h_3-h_2)}{N+1}\right\}+\frac{E_2\left(h_3{}^2-h_2{}^2\right)}{2}\right]+\frac{E\left(h_4{}^2-h_3{}^2\right)}{4(1+v_2)}$$

$$D_{11}=D_{22}=\frac{E(h^3{}_2-h^3{}_1)}{3(1-v^2{}_1)}+\frac{1}{1-v_f{}^2}\left[(E_1-E_2)\left\{\frac{(h_3-h_2)^3}{N+3}+\frac{2h_2(h_3-h_2)^2}{N+2}+\frac{h^2{}_2(h_3-h_2)}{N+1}\right\}+\frac{E_2(h^3{}_3-h^3{}_2)}{6(1-v_f{}^2)}\right]+\frac{E(h^2{}_4-h^2{}_3)}{3(1-v^2{}_2)}$$

$$D_{12}=\frac{vE\left(h_2{}^3-h_1{}^3\right)}{3\left(1-v_1{}^2\right)}+\frac{\left(E_1-E_2\right)v_f}{\left(1-v_f{}^2\right)}\left[\frac{\left(h_3-h_2\right)^3}{N+3}+\frac{2h_2\left(h_3-h_2\right)^2}{N+2}+\frac{h_2{}^2\left(h_3-h_2\right)}{N+1}\right]+\frac{\left(h_3{}^3-h_2{}^3\right)v_f}{6(1-v_f{}^2)}E_2+\frac{v\left(h_4{}^2-h_3{}^2\right)E}{3\left(1-v_2{}^2\right)}$$

$$D_{66}=\frac{E\left(h_2{}^3-h_1{}^3\right)}{6\left(1+v_1\right)}+\frac{\left(E_1-E_2\right)}{2\left(1+v_f\right)}\left[\frac{\left(h_3-h_2\right)^3}{N+3}+\frac{2h_2\left(h_3-h_2\right)^2}{N+2}+\frac{h_2{}^2\left(h_3-h_2\right)}{N+1}\right]+\frac{\left(h_3{}^3-h_2{}^3\right)}{6\left(1+v_f\right)}E_2+\frac{\left(h_4{}^2-h_3{}^2\right)E}{6\left(1+v_2\right)}$$

1.9 Result and discussion

The comparison of values of non-dimensional frequency parameters$\Omega=\omega R\sqrt{(1-v^2)\rho/E}$, for simply supported boundary conditions for homogeneous cylindrical shell with those of Loy *et al.* **[4]** is composed in Table 1. The present case was solved by the Raleigh-Ritz method while the frequency parameters in Loy *et.al*. **[4]** were obtained by the differential quadrature method. This comparison shows that the present results are nearly equal with each other. At *n*=2, the frequency parameter has the lowest value.

Table 1, Comparison of frequency parameters $\Omega=\omega R\sqrt{(1-v^2)\rho/E}$ for a cylindrical shell with simply supported-simply supported boundary conditions (*m*=1, *L/R*=20, *h/R*=0.01, *v*=0.3)

n	Loy *et al.* [4]	Present
1	0.016101	0.016101
2	0.009382	0.009363
3	0.022105	0.022085
4	0.042095	0.042075
5	0.068008	0.069788

The results frequencies (Hz) of vibration cylindrical shells having FGM are obtained. These cylindrical shells consist of two types of functionally graded material. Two materials: nickel and stainless steel are associated at inward and outward surfaces of a functionally graded cylindrical shell of 1st Type. While in 2nd Type they interchange their positions. The outer surface denoted by M_1 and inner denoted by M_2. Natural frequencies (Hz) of 1st Type and 2nd Type cylindrical shells are composed in Table 3 and 4 respectively for the half-axial wave mode *m* =1. Geometric parameters are mentioned in the Tables. Polynomial fraction law regulates the material distributions in FGM. The power law exponents are taken as: *N* A comparison of the result of natural frequencies (Hz) for a cylindrical shell for simply supported-simply supported edge conditions is given with the results of Warburton **[18]** in the Table 2. These boundary conditions are applied at the both end points of the cylindrical shell. The half-wave axial numbers are taken to be *m* = 1, 2, 3, 4, 5, 6 and the circumferential wave numbers are taken *n*=2, 3. From the comparison it observed that these results are close to each other.

= 0.5, 1, 15.The present obtained frequencies and those of Iqbal *et al*. **[13]** are compared with each other. The shell frequencies have been evaluated by the Raleigh - Ritz method and wave propagation method was applied by Iqbal *et al*. **[13]** to obtain them. The condition which is stated at both the ends is simply supported-simply supported. So the compared results coincided with each other

Table 3, Natural frequencies (Hz) comparisons of 1st Type cylindrical shells having simply supported – simply supported end condition (*m*=1, *L/R*=20, *h/R*=0.002)

	Iqbal *et al*. [13]			**Present**		
n	*N*=0.5	*N*=1	*N*=15	*N*=0.5	*N*=1	*N*=15
1	13.321	13.211	12.933	13.321	13.211	12.932
2	4.5168	4.480	4.3834	4.5195	4.4831	4.3858
3	4.1911	4.1569	4.0653	4.2014	4.1685	4.0788
4	7.0972	7.0384	6.8856	7.113	7.0563	6.9091
5	11.336	11.241	10.999	11.356	11.265	11.032

Table 2, Comparison of natural frequencies (Hz) for a simply supported- simply supported isotropic cylindrical shell ($L=8$in, $h=0.1$in, $v=0.3$,$\rho=7.35\times10^{-4}$Ibfs^2 in^{-4}, E=30×10^{6}Ibf in^{-2})

n	***N***	**Warburton[18]**	**Present**
2	1	2946.8	2042.7
	2	5637.8	5631.9
	3	8935.3	8926.4
	4	11405	1139.3
	5	13245	13243.7
3	1	2199.3	2194.4
	2	4041.9	4031.2
	3	6620.0	6605.9
	4	9124.0	9108.4
	5	11357	11343.4

Table 4, Natural frequencies (Hz) comparisons of 2nd Type cylindrical shells having simply supported-simply supported end condition. (*m*=1, *L/R*=20, *h/R*=0.002)

	Iqbal *et al*. [13]			**Present**		
n	*N*=0.5	*N*=1	*N*=15	*N*=0.5	*N*=1	*N*=15
1	13.103	13.211	13.505	13.102	13.209	13.504

2	4.4382	4.4742	4.5759	4.4386	4.4742	4.5767
3	4.1152	4.1486	4.2451	4.1256	4.1578	4.2522
4	6.9754	7.0330	7.1943	6.9945	7.0497	7.2051
5	11.145	11.238	11.494	11.172	11.2611	11.5070

From the above comparisons, it is clear that the present numerical procedure is efficient and valid and yields accurate results. Natural frequencies (Hz) for the present configurations of three layered cylindrical shells are furnished with variations depending on circumferential wave number, *n*, axial wave numbers, *m* and geometrical parameters. The end conditions considered here are simply supported – simply supported (SS-SS), clamped-clamped (C-C), clamped- free (C-F) and clamped- simply supported (C-SS). The three volume fraction laws: (i.) polynomial, (ii.) exponential and (iii.) trigonometric are applied to measure the material composition of functionally graded layer.

Table 5, Variation of frequencies of cylindrical shells with simply supported-simply supported edge conditions for 1st Type versus *n* (*h*=0.002, *L*=20, *R*=1)

	Polynomial	**Exponential**	**Trigonometric**	**Polynomial**	**Exponential**	**Trigonometric**
N	*N*=0.5			*N*=15		
1	13.5748	13.3747	12.7887	13.3356	13.1472	12.5716
2	4.4803	3.3600	4.2153	4.3893	3.2996	4.1429
3	3.4889	1.2446	3.2995	3.3233	1.2183	3.2431
4	5.5134	4.1349	5.2247	5.1750	4.0681	5.1371
5	8.7339	7.6059	8.2760	8.1820	7.4847	8.1382

From the Table 5, it is observed that the frequency obtained by using the polynomial volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 1st Type cylindrical shell with simply supported- simply supported boundary condition

Table 6, Variation of frequencies of cylindrical shells with clamped-clamped edge condition for 1st Type versus *n* (*h*=0.002, *L*=20, *R*=1)

	Polynomial	Exponential	Trigonometric	Polynomial	Exponential	Trigonometric
n	*N*=0.5			*N*=15		
1	22.3255	22.3171	21.1528	21.7079	21.7211	20.5885
2	7.5297	7.5349	7.0933	7.3182	7.3340	6.9042
3	4.1355	4.0913	3.8996	3.9749	3.9833	3.7960
4	4.4917	4.3080	4.2520	4.2090	4.1967	4.1399
5	6.5838	6.2462	6.2383	6.1177	6.0863	6.0741

From the Table 6, it is observed that the frequency obtained by using the polynomial volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 1st Type cylindrical shell with clamped-clamped boundary condition.

Table 7, Variation of frequencies of cylindrical shells with clamped-free edge condition for 1st Type versus *n* (h=0.002, L=20, R=1)

	Polynomial	Exponential	Trigonometric	Polynomial	Exponential	Trigonometric
N	*N*=0.5			*N*=15		
1	22.3256	22.2350	21.1529	21.7079	21.6407	20.5885
2	7.5302	7.1317	7.0938	7.3184	6.9392	6.9043
3	4.1365	3.1758	3.9006	3.9753	3.0863	3.7962
4	4.4927	3.4062	4.2529	4.2094	3.3139	4.1401
5	6.5845	5.6490	6.2389	6.1180	5.5018	6.0743

From the Table 7, it is observed that the frequency obtained by using the polynomial volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 1st Type cylindrical shell with clamped-free boundary condition.

Table 8, Variation of frequencies of cylindrical shells with clamped-simply supported edge condition for 1st Type versus *n* (h=0.002, L=20, R=1)

	Polynomial	Exponential	Trigonometric	Polynomial	Exponential	Trigonometric
N	*N*=0.5			*N*=15		
1	10.0573	9.9054	9.4714	9.7789	9.6408	9.2187
2	3.3181	2.4884	3.1359	3.2186	2.4195	3.0380
3	2.5839	0.9217	2.4436	2.4370	0.8933	2.3781
4	4.0833	3.0623	3.8694	3.7948	2.9831	3.7670
5	6.4683	5.6330	6.1293	5.9998	5.4885	5.9677

From the Table 8, it is observed that the frequency obtained by using the polynomial volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 1st Type cylindrical shell with clamped-simply supported boundary condition.

Table 9, Variation of frequencies of cylindrical shells with simply supported- simply supported edge condition for 2nd Type versus *n* (*h*=0.002, *L=20*, *R*=1)

From the

	Polynomial	Exponential	Trigonometric	Polynomial	Exponential	Trigonometric
n	*N*=0.5			*N*=15		
1	13.442	13.249	12.720	13.691	13.481	12.943
2	4.4217	3.3257	4.1903	4.5040	3.3875	4.2644
3	3.3347	1.2283	3.2690	3.3935	1.2552	3.3267
4	5.1848	4.0992	5.1720	5.2718	4.1675	5.2614
5	8.1969	7.5412	8.1931	8.3330	7.6656	8.3033

Table 9, it is observed that the frequency obtained by using the polynomial volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 2nd Type cylindrical shell with simply supported-simply supported boundary condition.

Table 10, Variation of frequencies of cylindrical shells with clamped-clamped edge condition for 2nd Type versus *n* (*h*=0.002, *L*=20, *R*=1)

From the

	Exponential	Polynomial	Trigonometric	Exponential	Polynomial	Trigonometric
n	*N*=0.5			*N*=15		
1	21.959	21.950	20.896	22.589	22.588	21.481
2	7.4147	7.4003	7.0094	7.6650	7.6153	7.2057
3	4.0281	4.0151	3.8502	4.2069	4.1299	3.9577
4	4.2461	4.2403	4.1898	4.4279	4.3569	4.3059
5	6.1591	6.1575	6.1432	6.3719	6.3247	6.3130

Table 10, it is observed that the frequency obtained by using the exponential volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 2nd Type cylindrical shell with clamped-clamped boundary condition.

	Polynomial	Exponential	Trigonometric	Polynomial	Exponential	Trigonometric
n	*N*=0.5			*N*=15		
1	21.9503	21.8769	20.8955	22.5883	22.4913	21.4812
2	7.4003	7.0099	7.0089	7.6153	7.2091	7.2056
3	4.0151	3.1066	3.8492	4.1299	3.1991	3.9575
4	4.2403	3.3397	4.1889	4.3569	3.4353	4.3057
5	6.1575	5.5596	6.1426	6.3246	5.1721	6.3129

Table 11, Variation of frequencies of cylindrical shells with clamped-free edge condition for 2nd Type versus *n* (*h*=0.002, *L*=20, *R*=1)

From the Table 11, it is observed that the frequency obtained by using the polynomial volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 2nd Type cylindrical shell with clamped-free boundary condition.

Table 12, Variation of frequencies of cylindrical shells with clamped-simply supported edge condition for 2nd Type versus *n* (*h*=0.002, *L*=20, *R*=1)

	Polynomial	**Exponential**	**Trigonometric**	**Polynomial**	**Exponential**	**Trigonometric**
n	*N*=0.5			*N*=15		
1	9.8997	9.7576	9.3688	10.1876	10.0316	9.6314
2	3.2569	2.4498	3.0861	3.3518	2.5217	3.1732
3	2.4573	0.9082	2.4076	2.5265	0.9377	2.4754
4	3.8202	3.0211	3.8091	3.9245	3.1032	3.9151

From the Table 12, it is observed that the frequency obtained by using the polynomial volume fraction law is the highest than those corresponding frequencies for other two volume fraction laws for 2nd Type cylindrical shell with clamped-simply supported boundary condition.

1.8 Conclusions

The vibration of cylindrical shells with FGM express by using the Raleigh-Ritz technique in this method. Three volume fraction laws are used to define the middle layer of tri-layer cylindrical shells. Two types of cylindrical shells are discussed in this method. The middle layer of cylindrical shell is FG which is composition of two materials Nickel and Stainless steel. At the inward surface of shell Stainless steel attached, while Nickel is attached at outward surface in 1st Type of shells. The position of these materials will interchange in 2nd Type. The results for simply-supported-simply supported, clamped-clamped, clamped-free and clamped-simply supported boundary conditions are obtained by this method. Following results are obtained by this present shell problem.

I. Circumferential wave number affect on the natural frequencies (Hz) of both Types of cylindrical shells. The frequencies increased and decreased by them.
II. Comparison of present obtained results with exponent power law for three volume fraction laws with the results of Loy *et al*.**[4]** and Naeem *et al*. **[10-11]** shows that they are good agreement with each other.
III. It observe that in 1st Type of cylindrical shell frequency is increasing as *N* increase and in 2nd Type it decreasing when *N* increase, due to interchanging the materials M_1 and M_2.
IV. The comparison of frequencies values of three volume fraction laws give the result that the frequency of 1st Type cylindrical shell increasing by polynomial fraction law, while in 2nd Type the frequency of cylindrical shells with clamped-clamped boundary condition increased by exponential law and other with polynomial fraction law. The comparison of variations of frequencies estimated that the recent method is valid and accurate. The obtained results are very close to previous result.

1.9 References

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