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Understanding the Greenium between Green and Conventional Bonds: A Simulation Study

Abdelmoumine Traore

Abstract

This paper examines the pricing disparity between green and conventional bonds (the greenium), drawing on empirical research findings that have yielded mixed results. We introduce a mathematical model to elucidate the conditions under which green bonds may be priced differently from their conventional counterparts. Unlike previous studies primarily focused on firm-level characteristics, our model incorporates investors' prosocial attitudes, income levels, and risk preferences to derive market prices for green bonds. By considering both supply and demand dynamics, we pioneer an equilibrium-based approach to pricing, departing from the assumptions of traditional models like CAPM and Black-Scholes. Additionally, we integrate regulatory risk into our analysis, introducing the concept of "green default" alongside pecuniary default. Our findings underscore the influence of investors' prosocial preferences, issuer environmental commitments, and issuance costs on the greenium. Moreover, stringent environmental policies and advancements in green technology mitigate the likelihood of green default, thereby bolstering market demand for green bonds. While climate risk exerts downward pressure on bond prices overall, its impact on the greenium varies based on the relative reduction in the equilibrium price of green bonds compared to conventional bonds.

Keywords: Investment, green bond, conventional bond utility, optimization, risk aversion, environmental regulation, green default

1 Introduction

Climate change and its effects are accelerating everywhere in the world, with climate-related disasters piling up, season after season. We assist with more devastating fires in California, persistent drought in Nevada, record floodings and landslides in Trinidad and Tobago, and heat waves in the Middle East and North Africa region. The fundamental issue with tackling climate change is how to bring the profit-seeking private sector to finance the environment, which is a public good (WB, 2020). The atmosphere is a global public good, with benefits that accrue to all, making private bargaining solutions unfeasible without interventions. Identifying and agreeing on policies for internalization of the social costs of Greenhouse Gas (GHG) emissions at the global level are extremely difficult, given the cost to some individuals and firms and the difficulties of global enforcement of such policies (Tirole, 2008). The public good problem is especially notable in environmental economics, which largely deals with analyzing and finding solutions to externality-related issues. Externalities pose fundamental economic policy problems when individuals, households, and firms do not internalize their economic transactions' indirect costs or benefits (Ward and

Sandler, 1986). Several innovative financial solutions have been created within the sustainable debt universe to correct private markets' failure to finance green projects. Among these solutions, green bonds are the most issued sustainable debt instruments (49.5% of share) with a cumulative market size of $1 trillion (CBI, 2020).

This essay will focus on corporate green bonds, particularly the greenium, defined as the price differential between a green bond and a strictly conventional bond of similar characteristics (face value, coupon, and maturity). According to the International Market Capital Association (ICMA, 2021), green bonds are any bond instrument where the proceeds or an equivalent amount will be applied to finance or re-finance, in part or full, new and existing eligible green projects. Here, a green project means a project that makes products, develops or uses technologies that positively impact the environment. Whereas a conventional project is not expected to generate a positive environmental impact. Green bonds distinguish themselves from conventional bonds according to three key salient features: (i) Green bonds finance green projects that have a proposed positive impact on the environment, (ii) there is an extra cost for the certification and the monitoring of the environmental performance of green bonds, (iii) green bonds attract environmentally aware investors who may be more willing to purchase green bonds relative to conventional bonds.

As of June 2021, IFC (International Finance Corporation) green bond proceeds have supported 236 green-eligible projects since 2014. The total committed amount for these projects is 9.4 billion USD, of which 7.7 billion USD has been disbursed (IFC Green Impact Report, 2020). For example, in 2021, IFC used the proceeds of the green bonds to finance the construction and operation of two mixed-use, modern, and energy-efficient office buildings in Iasi, Romania. The climate financing committed was $31.86 million. The project has made annual electricity savings of 2,656,916 KWh and is expected to reduce annually. GHG emission by 882 tons of CO2 (tCO_2) eq/year. The green bond proceeds were also used to finance the expansion of Nedbank's green portfolio in South Africa to increase access to climate finance through demonstration and capacity building initiatives, which will foster greater climate resilience in the South African banking sector by reducing Greenhouse Gas (GHG) emissions. The climate financing was $200 million and is expected to reduce GHG by 519,501 tCO2 eq/year.

The previous examples show that green bonds are sold with a promise to reduce carbon emissions. When any bond is purchased, there is a risk of financial default, which is the issuer's failure to pay the bond's monetary returns to the investor. In the case of a green bond, there is an additional risk that the green project fails to deliver its expected green benefits. Green default is specific to green bonds and occurs when the issuer fails to reach its promised environmental target due to the stochasticity of green effects[1], as well as the economic and regulatory environments. For example, green technology can become defective or obsolete during the bond's maturity. Solar panels do not always deliver at their full capacity. The risk of green default can affect the valuation of a green bond. A key difference between a green bond and a conventional bond is that the green bond issuance process requires due diligence in managing proceeds and reporting. In terms of costs, this is usually comprised of the verification and certification fees, as well as the internal staffing time to set up the proper protocols and systems to monitor green performance, e.g., the requirements of the Climate Bonds Standards (CBI, 2020). Moreover, green bonds require establishing methods for environmental auditing performance (Sustainable Bond Insight, 2021).

In 2020, the total of climate-aligned bond issuance amounted globally to 390 billion USD taking into account the labeled green bonds (280 million USD) and the unlabelled climate-aligned bonds (110 billion USD).[2](CBI & DBS, 2021). This represents about 3.6 % of global bond issuance. Moreover, according to the World Bank, the worldwide private annual investment needed to finance climate action in the next 15 years is around 6 trillion USD per year. These facts show the state of the failure of private markets to finance green projects, as there is still an enormous investment gap that needs to be filled (Amundi IFC, 2021).

Green bonds suffer from several characteristics that slow their adoption by the private market. First, the green bond market may be associated with negative externalities: the reputational risk in case of suspicions of misusing the proceeds can be strong enough to make an issuer and an underwriter refrain from issuing a green bond. Second, the green bond market may be viewed as unstable and has imperfect information: The price benefit of green bonds remains relatively unclear for some issuers and buyers. There is a perception of an extra cost associated with a green issuance. Moreover, the impact of climate-related risks remains unclear for both the issuer and the investors. Third, the green bond market is affected by an absence of standardized legislation because countries are still deciding on what constitutes a green project. There is no universally accepted legal and commercial definition of a green bond due to the diversity of taxonomies and green standards. The EU moved first toward adopting a standard classification system for what constitutes a green bond, and the United States Securities and Exchange Commission (the SEC) is also developing its standards. There is also uncertainty regarding the actions that need

[1]There is no doubt about the viability of the green bond; in this essay, we assume that the green bond issuer is not conveying a false impression or providing misleading information about how a bond is environmentally friendly (greenwashing)

[2]Unlabelled climate-aligned bonds are not explicitly As green bonds by the issuer but finance climate change solutions

to be taken to penalize the issuers and compensate the investors in case of defaults in using proceeds.

The notion that green bonds may be differently valued is suggested by the fact that green bonds may attract a new niche of investors that are environmentally aware and motivated to pay a premium for the ”green” attributes of a green bond. According to a study by Harvard Business Review, green bond issuers attract a 21% larger share of long-term investors (the share of long-term investors increases from 7.1% to 8.6%), coupled with an increased share of green investors[3] (the share of green investors increases from 3% to 7%) (Flammer 2018, 2020). Almost all issuers (91%) in the CBI Green Bond Treasury Survey perceived that green bonds involved more engagement with investors compared to conventional bonds. Finally, environmentally aware investors may have different preferences than hedonic investors, which could affect the valuation of green bonds with respect to conventional bonds.

There exists empirical evidence that the price differential of green bonds remains relatively unclear for some issuers in both primary and secondary markets[4]. Some studies argue for significant positive price differences between green and similar ordinary bonds (Ehlers and Packer, 2017; Baker et al., 2018; Partridge and Medda , 2018; Hachenberg and Schiereck, 2018; Zerbib, 2018; Zerbib, 2019; Kapraun and Scheins, 2019; Gianfrate and Peri , 2019; Fatica et al., 2020). Others support a negative price differential between green and ordinary bonds (Karpf and Mandel, 2017; Bachelet et al., 2019). Another group of studies argues that there is no difference in price between green and conventional bonds (Reed et al., 2017; Larcker and Watts, 2019). Empirical studies are limited because of the non-homogeneity of time, sample size, bond profile, control variables, control measures for liquidity and maturity, methodologies, and statistical analysis techniques. Notably, each study employs its robustness checks; however, using individual methodologies and control variables on similar datasets yield conflicting results. These studies have relied on some form of regression and matched-pair analysis. However, even the municipal bond studies still experience small sample sizes, largely because the green municipal bond market in the US only started in 2013, and its issuance label has remained relatively low compared to the overall market. For example, in the US municipal market, pricing is highly sensitive to tax features (Atwood, 2003). Despite similar data sets, Karpf finds a positive greenium on US municipal bonds (Karpf and Mandel, 2018), whereas Zerbib and Baker, drawing from moderately different methodologies and control variables, find a negative greenium (Baker et al., 2018; Zerbib, 2019). Because of the ambiguity over what constitutes a Green Bond (GB), studies generate misleading yield estimates on the green premium since they employ different datasets, particularly for datasets taken before 2013 and the establishment of more robust GB taxonomies and databases. For example Nanayakkara and Colombage (2019) employ a large dataset with worldwide coverage (25 countries) of GBs, whereas Agliardi and Agliardi (2019) focus on an in-depth review of a single corporate bond.

The empirical literature shows that evidence of a greenium is ambiguous. Therefore, we aim to build a theoretical model to understand how the greenium varies.

[3] A green investor *i*s an investor with a high degree of preference for projects reducing carbon emissions.

[4] The primary market is where securities are created and investors can directly buy them from the issuing company, while the secondary market is where investors trade securities

Several theoretical models on conventional and green bond pricing currently exist in the literature. The capital asset pricing model and the option pricing theory are two of the best-known and most important developments in conventional bond pricing. The first model was provided by William Sharpe (1964), followed by Tobin (1958), Treynor (1965), Lintner (1965), and Mossin (1966), and all of them are indebted to the Markowitz (1952, 1959) portfolio model. The model assumes investors are risk averse and, when choosing among portfolios, care only about the mean and variance of their one-period investment return. As a result, investors choose ”mean- variance-efficient” portfolios, in the sense that the portfolios (i) minimize the variance of portfolio return, given expected return, and (ii) maximize expected return, given variance. Thus, the Markowitz approach is often called a ”mean- variance model.” However, this approach has several critiques: (i) The model is based on the assumption that borrowing and lending are at a risk-free rate, which is the same for all investors and does not depend on the amount borrowed or lent. (ii) The bond price depends only on the mean and variance of the bond's return and the investor's risk aversion. (iii) The CAPM model does not take into account the investor's prosocial preferences for the environment, the environmental performance of the green bonds, and (iv) the characteristics of the firm issuing the bonds are not taken into account by the CAPM model.

Fama and Eugene (2005) and Wurgler (2018) have attempted to resolve the third critique of the original CAPM model by incorporating the noneconomic motives of investors, such as environmental preferences. Fama and Kenneth (2005) use the market equilibrium approach to frame the price effects of disagreement and tastes. They have identified the general factors that determine the price effects of tastes for assets as consumption goods and found that distortions of expected returns can be large when (i) investors with asset tastes account for substantial invested wealth, (ii) they have tastes for a wide range of assets, (iii) they take positions much different from those of the market portfolio, and (iv) the returns on the assets they underweight are not highly correlated with the returns on the assets they overweight. Wurgler et al. (2018) start with a relatively standard asset pricing framework to understand how a clientele with a preference for green bonds, or more generically for any non-financial objective, affects prices and portfolio choice. Wurgler et al. (2018) found from their model that securities with positive environmental scores (such as green bonds) have lower expected returns (higher prices). The model of Wurgler et al. (2018) can only explain the positive greenium.

The option pricing theory, on the other hand, derives from the seminal paper of Black and Scholes (1973), in which an arbitrage argument is developed to solve the old problem of pricing option contracts in a completely new way. Conventional bond pricing theory was written by Black and Scholes (1973)/ Merton (1974), Leland and Toft (1996), and the Brits and de Earenne (1997) do not take into account the three key salient features of green bonds. This mathematical model is often taken to be the geometric Brownian motion, which describes the instantaneous change in the asset price as the product of the risk-free interest rate and volatility. Despite the Black Scholes model's popularity and widespread use, the model is built on some non-real-life assumptions (Yalincak et Orhun, 2005): (i) The Black Scholes theorem assumes assets move in a manner referred to as a random walk; random walk means that at any given moment in time, the price of the underlying asset can go up or down with the same probability. However, this assumption does not hold, as asset prices are determined by many factors that cannot be assigned the same probability in how they will affect the movement of stock prices. (ii) The model assumes constant volatility. While volatility can be relatively constant in very short-term periods, it is never constant in the long term. (iii) The model assumes that interest rates are constant and known. This assumption is also unrealistic. The model uses the risk-free rate to represent this constant and known rate. In reality, treasury rates can change in times of increased volatility. (iv) The pricing of assets using Black-Scholes does not consider the salient features of green bonds or the investor's characteristics, such as his risk aversion, prosocial attitude, and income level.

Building upon Merton's work, Agliardi et al. (2019) derive an expression for the green bond value depending on several factors, including asset volatility, tax rates, the effectiveness of the green technology, and a parameter measuring the sustainability advantage of the firm. They show that the greenium is increased if asset volatility increases, the parameters governing the green technology and the sustainability advantage increase, and corporate tax rates are decreased. They also show how an improvement in credit quality induced by the green label ultimately leads to a lower cost of capital for green bond issuers. Agliardi et al. (2021) developed a model for defaultable bonds incorporating stochastics about corporate earnings and stochastics due to climate-related risks, which determine downward jumps in the firm value. In particular, they study how bond pricing is affected by transition risks, such as those coming from an abrupt change in climate policies. They show how the issuer's credit quality changes due to its engagement in projects funded by green bonds. Green bonds improve the issuer's creditworthiness because they fund projects less hit by climate change policies. Agliardi et al. (2021) wrote a third article in which they introduce two sources of risk regarding the cash flows of the firm and the effectiveness of the financed green projects. They show how green bonds affect the issuer's creditworthiness, depending on the green project's correlation with the firm's core business. Although the models of Agliardi et al. (2019, 2021) include environmental parameters, they still suffer from the unrealistic assumptions of the Merton and Black-Scholes model discussed in the previous section.

All simulation papers on the pricing of green bonds, based on the CAPM and the Black Scholes models, are made on unrealistic assumptions (constant volatility, constant interest rate for borrowing and lending, Brownian motion). Papers based on the CAPM derive bond pricing based on the bond returns and investor characteristics (risk aversion, investor environmental preferences) (Fama and Eugene, 2005; Wurgler et al., 2018). Whereas papers based on the Black-Scholes theory develop bond pricing based on the firm's characteristics only (volatility, interest rate, green technology effectiveness, sustainability advantage) (Agliardi et al., 2019,2020,2021).

In this essay, we model the market price of green bonds at issuance as the result of the supply (the firm) and demand (investors) forces for these assets. Therefore, we are the first to model the price of green bonds as determined by an equilibrium of the supply of green bonds and the demand for green bonds. We develop a pricing theory not based on the unrealistic assumptions of the CAPM and Black-Scholes models. More importantly, unlike the empirical papers, our theory and simulation approach is as general as possible and is not sample-dependent, location-dependent, or time-dependent. This essay is the first to consider two types of default: pecuniary default and green default. Defaults are due to different sources of risks. This essay considers climate risk and environmental regulation through input and output prices. Concerning climate risk, weather events in the short run are considered. However, in the long run, climate change has been viewed as increasing and changing those risks. While long-term perspectives are considered in specifications, changes in weather-related risk due to climate change will be considered only within the context of changes in weather events in general. No specific link to climate change is explicitly considered. Compared to the past literature, this essay is the first to characterize green default as the failure of a green bond issuer to fulfill his promise to reach a given environmental target at maturity. We model the impact of green default on the pricing of green bonds. The paper will be divided into five (5) parts: (i) The elaboration of a theoretical model to estimate the equilibrium price of a conventional bond under investor risk neutrality. (ii) The elaboration of a theoretical model to estimate the equilibrium price of a conventional bond under investor risk aversion. (iii)The development of green bond pricing theory by introducing the concept of green default and certification costs within the conventional bond pricing theory. (iv) We derive the greenium as the differential between the market price of a conventional bond vs. an equivalent green bond at issuance. (v) We simulate to understand how the greenium is affected by climate risk, regulatory risk, technological efficiency, and the prosocial attitude of the investors.

This theory is relevant for policy-making decisions as it allows issuers to understand the conditions under which green bonds are valued higher than conventional bonds, i.e., conditions under which it is more profitable to issue a green vs. a conventional bond or vice-versa. Many firms want to emit bonds to finance their environmentally friendly projects; however, they do not know when to issue a green bond vs. a conventional bond. Another contribution of this essay is that we can quantify the effect of green default on the value of green bonds. Therefore, our results will help incentivize policymakers to finance research on green technology and regulate the economic and regulatory environment to reduce the probability of green default.

2 Conventional Bond Pricing Theory

In this section, we derive the price at issuance of a conventional bond issued by a firm as an equilibrium between the supply and the demand for the conventional bonds issued by that firm. To achieve this goal, (i) we derive the optimal number of bonds the firm supplies given stochastic factors (Price, input, and weather). (ii) We derive the optimal demand for the conventional bond considering risk-neutral investors and (iii) considering risk-averse investors. The challenges of this problem follow the multinomial nature of demand across the universe of bonds. Further, the bond issuing firm is faced with price and productivity uncertainty.

2.1 Assumptions

(i) Let O be a set of L heterogeneous firms defined as follows:

$$O = \{l_0, l_1, l_2, l_3, ..., l_L\}. \quad (1)$$

Let t be a production time interval. We consider a firm l producing output vector $Q_{l,t}$ using input vector $X_{l,t}$ during period t. $Q_{l,t}$ is a 1×M vector and $X_{l,t}$ is a 1×J vector. $Q_{l,t}$ is priced at P_t where P_t is a 1×M vector, and inputs are priced at R_t where R_t is a 1×J vector. Both P_t and R_t are stochastic. The subjective expectations of P_t and R_t as held by firm l are defined as $E_l(P_t)$, and $E_l(R_t)$.

(ii) Firm l is seeking to invest in technology by emitting a conventional bond with coupon c, face value Y, and maturity T. At the time of issuance, firm l is confronted with L-1 firms in the market that have already emitted similar bonds. F is the output function linking $Q_{l,t}$ to $X_{l,t}$ given the stochastic weather factors (τ_t) affecting the productivity or the efficiency of firm l and the flow of fixed factor services available to the firm represented by vector $\vartheta_{l,t}$. τ_t is also stochastic. The subjective[5] expectation of τ_t as held by firm l is defined as $E_l(\tau_t)$.

(iii) A bond emitted by firm l differs from a bond emitted by another firm because firms are heterogeneous by nature. A bond emitted by firm l can be treated as different, where $b_{l,0}$ is a differentiated price[6] at which firm l can sell n_l units of the emitted bond given the multinomial, heterogeneous nature of demand.

(iv) In this theory, we assume that the producer makes a single investment which occurs at time t=0 ($k_{l,0}$) ($\forall$ t > 0, $k_{l,t}$ =0). We assume that the fixed stock of investment ($K_{l,t}$) of the firm deteriorates[7] over time by α. The equipment purchased by the firm has a life of T_0, which is the time recommended by the manufacturer before replacement. It is important to mention that T_0 could be lower or higher than the bond maturity T. At T_0, the stock of investment K_{l,T_0} is equal to the salvage value of the initial investment ($k_{l,0}$). In this essay, we assume that K_{l,T_0} is equal to a certain percentage $\psi \in [0,1]$ of the value of the initial investment ($k_{l,0}$). In mathematical form, the above discussion can be rewritten as follows:

$$\begin{aligned} K_{l,0} &= 0, \\ K_{l,t} &= K_{l,t-1}(1-\alpha) + k_{l,t} \text{ for } t > 0. \end{aligned} \tag{2}$$

Since a single investment is made which occurs at time t=0, the sequence in (2) can be rewritten as:

$$\begin{aligned} K_{l,0} &= 0, \\ K_{l,t} &= k_{l,0}(1-\alpha)^{t-1} \text{ for } t > 0. \end{aligned} \tag{3}$$

Since at t = T_0 the stock of investment is equal to a certain percentage of the initial investment value ($\psi * k_{l,0}$), α can be computed using this terminal condition:

$$\begin{aligned} K_{l,T_0} &= \psi * k_{l,0}, \\ k_{l,0}(1-\alpha)^{T_0-1} &= \psi * k_{l,0}, \\ \alpha &= 1 - (\psi)^{\frac{1}{T_0-1}}. \end{aligned} \tag{4}$$

[5]In analyses of decision making, the subjective expected value represents the extent to which an outcome is (a) desired or valued and (b) thought to be probable by the decision maker. The choice of one alternative over others is to a considerable extent a function of the personal (or subjective) value placed by an individual on a specific act or outcome as well as the perceived probability (expectation) that the given alternative will lead to that outcome.

[6]Let $\bar{b}$ be the average price across all firms selling conventional bonds of similar characteristics to the type of bond emitted by firm l. In this essay, $b_{l,0}$ is the price of interest to us, not b

[7]In this paper, we assume that $K_{l,t}$ deteriorates over time. However, we could have assumed that $K_{l,t}$ stays fixed but the service flow $\vartheta_{l,t}$ of $K_{l,t}$ deteriorates over time.

In eq. (4), given ψ and T_0, α was computed. Similarly, we note that given T_0 and α, ψ can be determined as in eq. (4).

(v) Since firm l issues a bond to finance the technological investment ($k_{l,0}$), the firm needs to choose the number of bonds n_l to offer at every level of price $b_{l,0}$.

(vi) The bond pricing problem is structured as follows: a) The firm has imperfect knowledge of existing price dispersion across product attributes, thus "estimates" the Bertrand-Nash type equilibrium price for its new bond issuance, prices it at $b_{l,0}$= $b_{l,0}(a_l)$, where a_l is the vector of attributes for the new bond. b) Given $b_{l,0}$, the firm chooses how many bonds to issue c) If the cost of bond issuance is lower than the cost of taking a loan, the firm issues n_l^*. d) Thereafter, the markets price the bond. We use a Bertrand-Nash style theory of that pricing. Given n^*_l, there exists demand for n_l conditioned by a_l the attributes of that bond.

2.2 The Multiple Output Production Function

The producer follows a time-intensive production process initiated at the beginning of time t and completed at the end of period t. Whether the producer is a conventional bond issuer or any other type of bond issuer, they produce a vector of proprietary outputs $Y_{l,t}$, and a vector of non-proprietary outputs $S_{l,t}$ using a vector of short-term input controls committed at the beginning of period t $X_{l,t}$. A simple interpretation of $S_{l,t}$ is a waste product; every production process generates some waste regardless of the type of bond issued by the firm. Let $Q_{l,t}$ be a vector containing $Y_{l,t}$ and $S_{l,t}$ such that $Q_{l,t} = (Y_{l,t}, S_{l,t})$.

In this theory, we define $S_{l,t}$ as the environmental output and $Y_{l,t}$ as the private output. Papers written on modeling multiple output technologies can be classified into two groups based on the approach to modeling environmental outputs. The first group of papers considers a multi-equation representation of polluting technology, while the second group adopts an alternative single-equation specification of the production process in the presence of public outputs. The multi-equation representation primarily attributed to Fernández et al.(2002, 2005), Forsund(2009), and Murty et al.(2012) rely on the more traditional multiplicative radial formulation of a system of a desirable technology and its accompanying undesirable by-production. In contrast, the single-equation approach usually formalizes polluting technology in the form of a function under the joint weak disposability of private and public outputs in the spirit of Chambers et al.(1996), Weaver(1996), Chung et al.(1997) and Fare et al.(2005). Let G be the production output possibility set such that:

$$G = \{(Q_{l,t}) : X_{l,t} \text{ can produce } Q_{l,t}\}. \tag{5}$$

As demonstrated by Fare et al. (2005), the directional distance function allows representing in a single equation of the joint production of multi-outputs using multi-inputs when some of the outputs are public. The distance function inherits its properties from the set G satisfying the following standard axioms: (i) The output set is compact for each input vector, (ii) The outputs are weakly disposable, (iii) Jointness is satisfied by G, (iv) Public and private outputs are null-joint. Figure 1 in Appendix shows an illustration of the directional distance function.

We then define the production function F over the set G relating $Q_{l,t}$ to $X_{l,t}$ given τ_t affecting the productivity or the efficiency of firm l and the flow of fixed factors available to the firm $\vartheta_{l,t}$. We assume that the firm is efficient. Therefore the symmetric representation of production is the following:

$$\begin{aligned} & F \equiv F(Q_{l,t}, X_{l,t} | \vartheta_{l,t}, \tau_t) = 0, \\ & F \equiv F(Y_{l,t}, S_{l,t}, X_{l,t} | \vartheta_{l,t}, \tau_t) = 0, \\ & \text{where } Q_t = (Y_{l,t}, S_{l,t}). \end{aligned} \tag{6}$$

The vector of service flow ($\vartheta_{l,t}$) from the fixed stock of investment ($K_{l,t}$) is a state variable at time t and represents the flow or the impact at time t of the stock of investment ($K_{l,t}$) on production:

$$\begin{aligned} & \vartheta_{l,t} = \vartheta_{l,t}(K_{l,t}), \\ & \frac{\partial \vartheta_{l,t}}{\partial K_{l,t}} > 0, \\ & \frac{\partial^2 \vartheta_{l,t}}{\partial K_{l,t}^2} < 0. \end{aligned} \tag{7}$$

Recall $K_{l,t}$ defined in eq. (3) depends on the deterioration rate α and the life of the investment T_0 as defined in eq. (4).

The stochastic nature of τ_t leads the producer to make their decision based on subjective perceptions of possible occurrences of τ_t. Like Weaver (1977), we define the provisional production function under risk neutrality using the expected production function of F, which is the expectation of the first-order Taylor series expansion of F when the firm is risk-neutral.

$$E(F) \equiv F(Y_{l,t}, S_{l,t}, X_{l,t} | \vartheta_{l,t}, E_l(\tau_t)) = 0. \tag{8}$$

If F is continuously differentiable, then the inverse function of F^{-1} exists. Let F^{-1} be the inverse function of F, then an asymmetric form of the production frontier can be written:

$$Y_{l,t} = F^{-1}(S_{l,t}, X_{l,t} | \vartheta_{l,t}, E_l(\tau_t)). \tag{9}$$

The impact of $E_l(\tau_t)$ on $Y_{l,t}$ depends on the application setting. For example, in the agricultural industry, $E_l(\tau_t)$ could represent temperature during summer time, then an increase in E(τ_t) will reduce the yield $Y_{l,t}$. Whereas, in the mining industry, if E(τ_t) represents precipitation, then an increase in E(τ_t) will reduce the mineral production $Y_{l,t}$. According to Fare et al. (2005), the directional output distance function inherits its properties from the output possibility set $G(X_{l,t})$. These properties include:

$$\frac{\partial F}{\partial Y_{l,t}} < 0. \tag{10}$$

$$\frac{\partial F}{\partial S_{l,t}} > 0. \tag{11}$$

$$\frac{\partial F}{\partial X_{l,t}} > 0. \tag{12}$$

The second-order conditions require that F be concave around $(Y_{l,t}, S_{l,t}) \in G(X_{l,t})$.

2.3 The Theory of Conventional Bond Supply under Risk Neutrality

We start by defining the short-run profit function of the risk-neutral firm without conventional bond issuance. In equation (13), P_t' and R_t' are the transpose of the vectors P_t and R_t:

$$\Pi^{sr}_{l,t} \equiv P_t' Y_{l,t} - R_t' X_{l,t}. \tag{13}$$

The firm's risk-neutral choice conditional on $k_{l,0}$, as assuming free disposal of $S_{l,t}$, consists in maximizing (13) given (8) by choosing the optimal $Y_{l,t}$ and $X_{l,t}$:

$$\begin{aligned} &\max_{Y_{l,t}, X_{l,t}} E_l(\Pi^{sr}_{l,t}), \\ &\text{such that } F(Y_{l,t}, S_{l,t}, X_{l,t} | \vartheta_{l,t}(k_{l,0}), E_l(\tau_t)) = 0. \end{aligned} \tag{14}$$

The maximization problem in (14) can be written in a Lagrangean form as follows:

$$\max_{Y_{l,t}, X_{l,t}, \lambda_1} L_{\lambda_1, l,t} = \max_{Y_{l,t}, X_{l,t}, \lambda_1} E_l(P_t)' Y_{l,t} - E_l(R_t)' X_{l,t} + \lambda_1 F(Y_{l,t}, S_{l,t}, X_{l,t} | \vartheta_{l,t}(k_{l,0}), E_l(\tau_t)),$$
$$\lambda_1 \in R. \tag{15}$$

Problem 2.15 is convex if F is affine. The output and input price expectations, as well as weather expectations, are held by the issuer. The First Order Conditions (FOC) for the problem (15) are the followings:

$$\begin{aligned} \frac{\partial L_{\lambda_1, l,t}}{\partial Y_{l,t,m}} &= E_l(P_{t,m}) + \lambda_1 \frac{\partial F}{\partial Y_{l,t,m}} = 0 \; ; \forall m = 1, 2, ..., M, \\ \frac{\partial L_{\lambda_1, l,t}}{\partial X_{l,t,j}} &= -E_l(R_{t,j}) + \lambda_1 \frac{\partial F}{\partial X_{l,t,j}} = 0 \; ; \forall j = 1, 2, ..., J. \end{aligned} \tag{16}$$

We recall that only M+J-1 of these equations in eq.(16) is independent because of the efficiency condition for the provisional production function that has to be met as well. Solving the system of equations using the M+J-1 independent equations in eq.(16) and the efficiency condition of the provisional production function (eq.(8)), we find the optimal outputs and inputs for the firm:

$$\begin{aligned} Y^*_{l,t} &= Y^*_{l,t}(\vartheta_{l,t}(k_{l,0}), E_l(P_t), E_l(R_t), E_l(\tau_t)), \\ X^*_{l,t} &= X^*_{l,t}(\vartheta_{l,t}(k_{l,0}), E_l(P_t), E_l(R_t), E_l(\tau_t)). \end{aligned} \tag{17}$$

$S^*_{l,t}$ and $Y^*_{l,t}$ are determined by the choices of $X^*_{l,t}$ and $k_{l,0}$. Therefore, $S^*_{l,t}$ can also be written in reduced form as:

$$S^*_{l,t} = S^*_{l,t}(\vartheta_{l,t}(k_{l,0}), E_l(P_t), E_l(R_t), E_l(\tau_t)). \tag{18}$$

Although the conventional bond issuer produces a vector of public outputs ($S^*_{l,t}$), they are not considering that in their contemporaneous profit calculations. Substituting (17) into (13), we derive the restricted short-run profit function of the risk-neutral firm:

$$\Pi^{sr,*}_{l,t} \equiv E_l(P_t)' Y^*_{l,t}(\vartheta_{l,t}(k_{l,0}), E_l(P_t), E_l(R_t), E_l(\tau_t)) - \dots$$
$$E_l(R_t)' X^*_{l,t}(\vartheta_{l,t}(k_{l,0}), E_l(P_t), E_l(R_t), E_l(\tau_t)), \qquad (19)$$
$$= \Pi^{sr,*}_{l,t}(\vartheta_{l,t}(k_{l,0}), E_l(P_t), E_l(R_t), E_l(\tau_t)).$$

The first derivatives of the restricted profit function with respect to input and output prices, as well as the flow of fixed factors ($\vartheta_{l,t}$) have the following signs:

$$\frac{\partial \Pi^{sr,*}_{l,t}}{\partial E_l(P_t)} = Y^*_{l,t} > 0 \text{ (Hotelling's Lemma)},$$
$$\frac{\partial \Pi^{sr,*}_{l,t}}{\partial E_l(R_t)} = -X^*_{l,t} < 0 \text{ (Hotelling's Lemma)}, \qquad (20)$$
$$\frac{\partial \Pi^{sr,*}_{l,t}}{\partial \vartheta_{l,t}} > 0.$$

Eq. (20) shows that the profit function satisfies Hotelling's Lemma, which asserts that there is a positive (negative) relationship between profit and output (input) prices. The higher the flow of fixed factors $\vartheta_{l,t}$, the higher the level of profit that the firm can obtain.

The second derivatives of the restricted profit function with respect to input and output prices, as well as the flow of fixed factors ($\vartheta_{l,t}$), have the following signs:

$$\frac{\partial^2 \Pi^{sr,*}_{l,t}}{\partial E_l(P_t)^2} = \frac{\partial Y^*_{l,t}}{\partial E_l(P_t)} > 0,$$
$$\frac{\partial^2 \Pi^{sr,*}_{l,t}}{\partial E_l(R_t)^2} = -\frac{\partial X^*_{l,t}}{\partial E_l(R_t)} > 0, \qquad (21)$$
$$\frac{\partial^2 \Pi^{sr,*}_{l,t}}{\partial \vartheta^2_{l,t}} < 0.$$

Eq.(21) shows that the second partial derivatives of profit with respect to output and input prices are positive because the optimal quantity produced by the firm increases with output prices. In contrast, the optimal input chosen by the firm decreases with input prices. The profit function exhibits diminishing marginal returns with respect to the flow of fixed factors.

When the firm issues a conventional bond, they need to pay an issuance cost as a function of the number of bonds issued, ie. $IC(n_l)$. Issuance costs are expenditures associated with underwriting and issuing debt securities and equity securities. Issuance costs include audit fees, investment banking fees, legal fees, marketing expenses, and Securities and Exchange Commission (SEC) registration fees (CFA institute, 2021). The cost of issuance is defined as follows:

$$\frac{\partial IC}{\partial n_l} < 0,$$
$$\frac{\partial^2 IC}{\partial n_l^2} > 0. \qquad (22)$$

The cost of issuance is such that the higher the quantity of bond emitted, the lower the cost of issuance for the firm. The cost function has a convex shape, so the negative of the issuance cost function has a concave shape.

We note that the issuer needs to make an intertemporal choice of $k_{l,0}$. We recall from the assumption section that T_0 is different from T, the bond maturity[8]. We can then define the long-run restricted profit function over the life of the investment T_0 as follows:

$$\begin{aligned} \Pi^{lr,*}_{l,t} &= n_l b_{l,0} + \Pi^{sr,*}_{l,t} - IC \text{ for } t = 0, \\ \Pi^{lr,*}_{l,t} &= \Pi^{sr,*}_{l,t} - n_l c\mathrm{Y} \text{ for } 1 \leq t < T, \\ \Pi^{lr,*}_{l,t} &= \Pi^{sr,*}_{l,t} - n_l c\mathrm{Y} - n_l \mathrm{Y} \text{ for } t = T, \\ \Pi^{lr,*}_{l,t} &= \Pi^{sr,*}_{l,t} \text{ for } T < t \leq T_0. \end{aligned} \tag{23}$$

In this essay, we assume that the issuance cost in the conventional market is negligible, so IC=0. Eq. (23) shows that when the firm emits a bond, it receives $n_l b_{l,0}$ at t=0 but needs to pay $n_l c$Y from period t =1 to t=T .Y represents the face value of the bond. At the bond maturity period, the firm pays the face value of the bonds to all the bondholders. Let $\delta_l \in (0,1)$ be the discount rate of the future profit values of firm l. From eq. (23), we can derive the net present value of the profit that will be obtained by the firm over the life of the investment T_0 as:

$$\begin{aligned} \Pi^{lr,*}_{l} &= NPV\left(\Pi^{lr,*}_{l,t}\right), \\ &= n_l b_l + \sum_{t=0}^{T_0} \delta_l^t \Pi^{sr,*}_{l,t} - \sum_{t=1}^{T} \delta_l^t n_l c\mathrm{Y} - \delta_l^T n_l \mathrm{Y}. \end{aligned} \tag{24}$$

We assume the amount of investment made by the firm must be equal to the amount of money received from the bond sale. We note that the face value of a bond (Y) might differ from the price $b_{l,0}$ at which the bond is sold at issuance. Additionally, we assume that the equilibrium price $b_{l,0}$ is not changing throughout the bond.

$$k_{l,0} = n_l b_{l,0}. \tag{25}$$

The net present value of the long-run profit function (eq. 24) can be equivalently written in terms of the level of investment $k_{l,0}$:

$$\Pi^{lr,*}_{l} = k_{l,0} + \sum_{t=0}^{T_0} \delta_l^t \Pi^{sr,*}_{l,t}\left(\vartheta_{l,t}(k_{l,0})\right) - \sum_{t=1}^{T} \delta_l^t \frac{k_{l,0}}{b_{l,0}} c\mathrm{Y} - \delta_l^T \frac{k_{l,0}}{b_{l,0}} \mathrm{Y}. \tag{26}$$

Let Ω_1 be the net cost of financing through bond issuance. This cost is calculated by estimating the amount of money that needs to be repaid by the firm to bondholders *vs* the amount it received ($k_{l,0}$) from bondholders. We define Ω_1 as:

$$\begin{aligned} \Omega_1 &\equiv \sum_{t=1}^{T} \delta_l^t n_l c\mathrm{Y} + \delta_l^T n_l \mathrm{Y} - n_l b_{l,0}, \\ \Omega_1 &\equiv \sum_{t=1}^{T} \delta_l^t \frac{k_{l,0}}{b_{l,0}} c\mathrm{Y} + \delta_l^T \frac{k_{l,0}}{b_{l,0}} \mathrm{Y} - k_{l,0}. \end{aligned} \tag{27}$$

[8]We assume that $b_{l,0}$ exists such that bond issuance is more profitable than bank borrowing

If the firm had taken a loan with a fixed rate ω, let Ω_2 be the net cost of bank debt financing. This cost is calculated by estimating the amount of money that needs to be repaid by the firm to the bank *vs* the amount it received ($k_{l,0}$) from the bank at t=0.

$$\Omega_2 \equiv \sum_{t=1}^{T} \delta_l^t k_{l,0}\omega - k_{l,0}(1 - \delta_l^T). \tag{28}$$

We assume that the firm issues bonds when $\Omega_1^* \leq \Omega_2^*$. This implies:

$$\begin{aligned} 0 &\leq \Omega_2^* - \Omega_1^*, \\ \sum_{t=1}^{T_0} \delta_l^t \frac{k_{l,0}^*}{b_{l,0}} c\Upsilon + \delta_l^T \frac{k_{l,0}^*}{b_{l,0}}\Upsilon &\leq \delta_l^T k_{l,0}^* + \sum_{t=1}^{T} \delta_l^t k_{l,0}^* \omega. \end{aligned} \tag{29}$$

The firm's choice of investment ($k_{l,0}^*$) is obtained by maximizing its long-run net discounted profit (eq. 26).

$$\max_{k_{l,0}} \Pi_l^{lr,*} = \max_{k_{l,0}} \left[\sum_{t=0}^{T_0} \delta_l^t \Pi_{l,t}^{sr,*}(\vartheta_{l,t}(k_{l,0})) + k_{l,0} - \sum_{t=1}^{T} \delta_l^t \frac{k_{l,0}}{b_{l,0}} c\Upsilon - \delta_l^T \frac{k_{l,0}}{b_{l,0}} \Upsilon \right]. \tag{30}$$

The firm solves the above unconstrained maximization problem to obtain the optimal level of investment $k_{l,0}^*$.

The second derivative $\frac{\partial^2 \Pi_l^{lr,*}}{\partial k_{l,0}^2} < 0$ because $\frac{\partial^2 \Pi_{l,t}^{sr,*}(\vartheta_{l,t}(k_{l,0}))}{\partial k_{l,0}^2} < 0$ as defined in eq.(21).

Therefore, the objective function is concave in $k_{l,0}$, which means that problem (30) is a convex optimization[9] problem. We are then guaranteed an optimal solution ($k_{l,0}^*$). The first order conditions (FOC) for problem (30) is the following:

$$\frac{\partial \Pi_l^{lr,*}}{\partial k_{l,0}} = \sum_{t=0}^{T_0} \delta_l^t \frac{\partial \Pi_{l,t}^{sr,*}(\vartheta_{l,t}(k_{l,0}))}{\partial k_{l,0}} - \sum_{t=1}^{T} \delta_l^t \frac{c\Upsilon}{b_{l,0}} - \delta_l^T \frac{\Upsilon}{b_{l,0}} + 1 = 0. \tag{31}$$

By solving eq.(31), we obtain the optimal investment made by the firm:

$$k_{l,0}^* = k_{l,0}^*(b_{l,0}, c, \Upsilon, E_l(P_t), E_l(R_t), E_l(\tau_t), \alpha, \delta_l, \omega, T, T_0, \Gamma_F). \tag{32}$$

Inequality 2.29 is then tested at $k_{l,0}^*$. If it is found that inequality 2.29 holds, then $k_{l,0}^*$ is selected. If it does not hold, then $k_{l,0}^* = 0$.

Knowing $k_{l,0}^*$, the optimal level of bonds that will be supplied by firm l can be obtained as follows:

$$\begin{aligned} n_l^{s,*} &\equiv \frac{k_{l,0}^*}{b_{l,0}}, \\ n_l^{s,*} &= n_l^{s,*}(b_{l,0}, c, \Upsilon, E_l(P_t), E_l(R_t), E_l(\tau_t), \alpha, \delta_l, \omega, T, T_0, \Gamma_F). \end{aligned} \tag{33}$$

Where $n_l^{s,*}$ is the optimal quantity of bonds supplied by firm l at price $b_{l,0}$. Γ_F are parameters defining the production function.

[9] Maximizing a concave function is equivalent to minimizing the negative of the concave function, which is a convex

2.4 The Theory of Investor Demand under Risk Neutrality

2.4.1 Characterization of the Conventional Bond Expected Yield

Bond yield is the pecuniary return that an investor i anticipates on a bond per dollar invested. Let $y_{i,l}$ be the expected yield of investor i for bond $b_{l,0}$ issued by firm l. In this essay, we compute $y_{i,l}$ by dividing the investor's expected pecuniary return ($W_{i,l}$) of the bond by the price of the bond ($b_{l,0}$):

$$y_{i,l} = \frac{W_{i,l}}{b_{l,0}}. \tag{34}$$

Let DF be the state that the firm defaults to pay the bondholders before the bond maturity, and let $\overline{DF}$ be the state that the firm does not default to pay the bondholders at any time during the bond maturity. The subjective expectations of P_t, R_t, τ_t as held by investor i are defined as $E_i(P_t)$, $E_i(R_t)$, and $E_i(\tau_t)$. We note that $E_i(P_t)$, $E_i(R_t)$, and $E_i(\tau_t)$ may be respectively different from $E_l(P_t)$, $E_l(R_t)$, and $E_l(\tau_t)$ as the issuer and the investor may not have the same subjective expectations about the stochastic variables.

To specify expected return $W_{i,l}$, we need to consider the return with the possibility that the firm commits pecuniary default. The probability of pecuniary default of the firm is composed of two parts: the systematic part and the idiosyncratic part (Weaver, 2001). The systematic part is the general market financial risk, and the idiosyncratic part is the risk of default specific to the firm l, which can be controlled through a portfolio of investment in assets (Eisenberg and Noe, 2001).

The Idiosyncratic probability of pecuniary default of the firm

Like other papers in the structural bond valuation literature, we characterize default as the first time the firm value crosses a default boundary κ. This approach originated with Black and Scholes (1973), Merton (1974), and Black and Cox (1976) and continues with Longstaff and Schwartz (1995) and, more recently, with Briys and Varenne (1997), Tauren (1999) and Collin-Dufresne and Goldstein (2001). In Black and Scholes (1973) and Merton (1974), all debts mature on the same day, and the firm defaults when its value is lower than the payment due. Hence, the default boundary κ is equal to the face value of the maturing debt. If default occurs, the claimants receive the liquidation value of the firm in order of priority. Agliardi et al. (2019, 2020, 2021) model bankruptcy as the time when the firm's post-investment equity value drops to zero for the first time and debt holders take over and obtain the firm's unlevered assets net of bankruptcy costs. We follow the approach of Black and Cox (1976), Longstaff and Schwartz (1995), and Barogne et Castagna (1998), who assume that the firm is forced into default by its debt covenants the first time its value falls below a constant threshold κ. In this case, κ can be viewed as the face value of the liabilities of a firm that has a constant dollar amount of debt outstanding at all times.

In our essay, we represent the value of firm l evaluated by investor i at a given time by its profit ($\Pi^{lr,*}_{i,l,t}$). $\Pi^{lr,*}_{i,l,t}$ is derived with the same methodology used by the issuer (sec. 3.2.3), except that δ_l, $E_l(P_t)$, $E_l(R_t)$, and $E_l(\tau_t)$ are respectively replaced by δ_i, $E_i(P_t)$, $E_i(R_t)$, and $E_i(\tau_t)$. We assume that the firm does not have any debt before issuing the bonds; therefore, the only debt of the firm is the periodic payment of the coupons to bondholders, which is equal to $\eta^*_{i,l}c\text{Y}$. In our essay, the boundary default threshold κ equals $n^*_{i,l}c\Upsilon$. $n^*_{i,l}$ is the optimal number of bonds supplied by firm l at price $b_{l,0}$ as estimated by investor i. Let $pd_{i,l}(\tilde{t})$ be the probability estimated by investor i that the firm issuing the bond $b_{l,0}$ defaults at time $\tilde{t} < T$. This probability can be written as follows:

$$pd_{i,l}(\tilde{t}) = P(\Pi^{lr,*}_{i,l,\tilde{t}} \leq n^*_{i,l}c\Upsilon \mid \Pi^{lr,*}_{i,l,t} > n^*_{i,l}c\Upsilon)\ \forall\ t < \tilde{t},$$
$$pd_{i,l}(\tilde{t}) = pd_{i,l}\left(\tilde{t}, b_{l,0}, c, \Upsilon, E_i(P_t), E_i(R_t), E_i(\tau_t), \alpha, \delta_l, \omega, T, T_0, \Gamma_F\right). \quad (35)$$

where $\Pi^{lr,*}_{i,l,\tilde{t}}$ is defined in equation (26) and evaluated at the optimal investment $k^*_{i,l,0}$. Eq. (35) shows that the idiosyncratic probability distribution of default at a given time depends on the stochastics affecting the profit of the firm as well as internal parameters specific to the firm, such as the parameters of the production function and the parameters defining the cost function of issuance. The idiosyncratic subjective probability of investor i that firm l defaults at any time before the bond maturity is:

$$PD_{i,l}(T) = 1 - P(\Pi^{lr,*}_{i,l,t} > n^*_{i,l}c\Upsilon),\ \forall\ t \in [1, T]. \quad (36)$$

This probability can be estimated by the researcher following the procedure by Jason Hsu et al. (2003). In reality, $PD_{i,l}(T)$ is difficult to estimate for a bond investor as they need to have access to private and sometimes confidential information about the firm, such as the number of inputs used by the firm, the different costs incurred by the firm, the production and cost functions of the firm. Moreover, the methodology for computing profit varies from one firm to another. In the next section, we compute the systematic risk of default of the firm.

The Systematic Risk of Default of the Firm

The systematic risk of default of the firm at any future time t is based on parameters[10] defining the distribution of the stochastic variables affecting the entire market at that time.

$$pd_{i,O}(t) \equiv pd_{i,O}(\Gamma_{Pt}, \Gamma_{Rt}, \Gamma_{\tau t})(t). \quad (37)$$

The probability of systematic default before the bond maturity ($PD_{i,O}(T)$) is computed as follows:

$$PD_{i,O}(T) = \prod_{t=0}^{t=T}(1 - pd_{i,O}(t)). \quad (38)$$

The Total Probability of Default of the Firm

$PD_{i,l}(T)$ (eq. 36) is the idiosyncratic probability of default of the firm before time T, and $PD_{i,O}(T)$ (eq. 38) is the systematic risk of default during the period of maturity of the bond, where O is the set representing the market. We define $\tilde{P}D_{i,l}(T)$ as the total probability of default of the firm taking into account both systemic and idiosyncratic risks.

$$\tilde{P}D_{i,l}(T) = MIN(PD_{i,O}(T) + PD_{i,l}(T), 1). \quad (39)$$

[10]Parameters could be the minimum, the maximum, the variance, or the mean of the stochastic variables

$W^{DF}_{i,l}$ is the expected return with a default, and the return without a default is $W^{\overline{DF}}_{i,l}$.

DF and $\overline{DF}$ are two mutually exclusive states because the firm cannot be in the two states (default and no default) at the same time. Also, if the firm is in state DF at time t, it does not prevent the firm from being in either state DF or $\overline{DF}$ at time t+1. Therefore, it is reasonable to assume that DF and $\overline{DF}$ are independent states. The modeling of bond pricing with a default requires estimating the probability that firm l defaults before T (the bond maturity). This probability is subjective as it depends on the expectations of investor i regarding prices, weather, and even the investor's assumptions regarding the future costs and revenue of the firm.

We assume that $W^{DF}_{i,l}$ and $W^{\overline{DF}}_{i,l}$ are independent since DF and $\overline{DF}$ are independent[11] and mutually exclusive events.

$$W_{i,l} = \widetilde{PD}_{i,l}(T)W^{\overline{DF}}_{i,l} + (1 - \widetilde{PD}_{i,l}(T))W^{DF}_{i,l}. \tag{40}$$

In the next sections, we derive expressions for $W^{\overline{DF}}_{i,l}$, $W^{DF}_{i,l}$.

Expected Return of the Conventional Bond Without Default

$\delta_i \in (0,1)$ represents the discount rate chosen by the investor for the future bond returns. The bondholder obtains the coupon payments up to maturity plus the face value of the bond at maturity.

$$W^{\overline{DF}}_{i,l} = \sum_{t=1}^{T} \delta_i^t (c\text{Y}) + \delta_i^T \text{Y} - b_{l,0}. \tag{41}$$

Expected Return of the Conventional Bond With Default

The expected return of the bond with default depends on the time the company issuing the bond will default. In this essay, we assume that the investor gets zero coupons/and revenue when the firm defaults and onwards. Suppose the investor expects the company issuing the bond to default at time $\tilde{t}$; we define the expected return at time $\tilde{t}$ as follows:

$$W_{i,l,\tilde{t}} = \sum_{t=1}^{\tilde{t}-1} e^{-\delta_i * t}(c\text{Y}) - b_{l,0}. \tag{42}$$

Eq. (41) shows that when the firm defaults at $\tilde{t}$, the investor stops receiving coupon payments at $\tilde{t} - 1$ and onwards. The average expected return with a default can be written as follows :

$$W^{DF}_{i,l} = \sum_{\tilde{t}=1}^{T} W_{i,l,\tilde{t}} * pd_{i,l}(\tilde{t}). \tag{43}$$

$PD_{i,l}(T)$ (eq. 36), $PD_{i,0}(T)$ (eq. 38), $W^{\overline{DF}}_{i,l}$ eq. (41), and $W^{DF}_{i,l}$ eq. (43) can be substituted in eq. (40) to calculate $W_{i,l}$. Once $W_{i,l}$ computed, it can be replaced in eq. (34) to obtain $y_{i,l}$.

[11] In case of dependence between $W^{DF}_{i,l}$ and $W^{\overline{DF}}_{i,l}$, we can can introduce a covariance term in eq. (33) to capture the variation between the returns in the two states.

2.4.2 The Individual Investor Choice to Buy a Conventional Bond vs. Alternative Bonds

Let's consider set B containing L elements corresponding to the number of firms issuing bonds in the representative market. B can be represented as follows:

$$B = \{B_0, B_0, B_2, B_3, ..., B_L\} \text{ for } l = 0, 1, ..., L - 1. \quad (44)$$

A risk-neutral investor i chooses to purchase bond B_0 over the alternative bonds if the investor's expected yield of bond B_0 is higher than its expected yield on any of the alternative bonds. B_0 is defined as a bond issued by firm l_0.

Following Train's (2009) multinomial choice approach, $y_{i,l}$ is defined as the systematic component of the actual yield $y^*_{i,l}$. $y_{i,l}$ is given by the theory and is observed by the researcher. (eq. 34). $\epsilon_{i,l}$ is a random yield; the researcher does not see this but knows its distribution. The actual yield ($y^*_{i,l}$) is realized by the agent but is partly captured by the researcher because of $\epsilon_{i,l}$. In mathematical form:

$$y^*_{i,l} = y_{i,l} + \epsilon_{i,l}. \quad (45)$$

The maximization problem involves comparing observable yields ($y_{i,l}$) while accounting for randomness ($\epsilon_{i,l}$). investor i chooses bond B_0 over L alternative bonds if two conditions are met:

$$\begin{array}{ll} \textbf{Condition 1} & y^*_{i,0} > y^*_{i,l} , \forall l \neq 0, \\ \textbf{Condition 2} & y^*_{i,0} > y^*_{i,free} . \end{array} \quad (46)$$

The second condition can be derived from the first condition using the financial literature approach of a risk-free asset. According to Cohn et al. (1975), the notion of the "riskless asset" has been important in developing modern capital market theory. Empirical work in finance, especially with respect to tests of capital market theory has often employed short-term Treasury Bills as a surrogate for the riskless asset. Following this approach, we define $\overline{B}$ as the set containing all bonds except the bond chosen by the investor that we denote as B_0.

$$\overline{B} = B \setminus \{B_0\}. \quad (47)$$

As discussed by Longtstaff (1993) on the valuation of options on coupon bonds, the value of the weighted sum of a portfolio of discount bonds can be expressed with the value of a riskless coupon bond. In financial literature, it is not uncommon to derive the Black-Scholes formula by introducing a continuously rebalanced risk-free portfolio containing an option and underlying stocks. With arbitrage, the return from such a portfolio will match returns on risk-free bonds (Merton, 1974). This property leads to the Black-Scholes partial differential equation satisfied by the arbitrage price of an option (Musiela et al., 2006; Baz et al., 2004). Friesen (1979) shows that the risk-free bond can be replicated by a portfolio of two Arrow-Debreu securities. This portfolio exactly matches the payoff of the risk-free bond. This is because if its price differed from the risk-free bond, we would have an economic arbitrage opportunity. Riskless profits can be made through some trading strategy when an arbitrage opportunity is present. In this specific case, if the portfolio of Arrow-Debreu securities differs in price from the price of the risk-free bond, then the arbitrage strategy consists in buying the lower-priced one and selling short the higher-priced one. Since each has the same payoff profile, this trade would leave us with zero net risk (the risk of one cancels the other's risk because we have bought and sold in equal quantities the same payoff profile) (Copelan et al., 2004; Breeden Douglas, 1978). Therefore, we obtain the following relationship between the portfolio of all alternatives ($\overline{B}$) and the risk-free asset (B_{free}):

$$\overline{B} \equiv B_{free}. \tag{48}$$

$$\Longrightarrow\ y^*_{i,\overline{B}} \equiv y^*_{i,free}. \tag{49}$$

Therefore, the multinomial choice of the investor to choose B_0 relative to alternative corporate bonds (Condition 1) implies that the investor must also prefer to choose B_0 compared to a risk-free bond (Condition 2). Condition 1 implies Condition 2, but the reverse is not always true.

The Probability of Satisfying Condition 1

Let $\Delta\epsilon_{i,l,0} = \epsilon_{i,l} - \epsilon_{i,0}$ and $\Delta y_{i,0,l} = y_{i,0} - y_{i,l}$. Condition 1 can be equivalently written as follows:

$$\begin{aligned} \epsilon_{i,l} - \epsilon_{i,0} &< y_{i,0} - y_{i,l},\ \forall l \neq 0, \\ \Delta\epsilon_{i,l,0} &< \Delta y_{i,0,l},\ \forall l \neq 0. \end{aligned} \tag{50}$$

Because of $\epsilon_{i,0}$'s and $\epsilon_{i,l}$'s, the decision of investor i to choose B_0 over all other firm bonds $b_{l,0}$ ($l \neq 0$) is a stochastic choice, from the researcher's point of view. Specific assumptions about the distribution of $\epsilon_{i,0}$'s will determine investor i's choice probabilities. Let h be the distribution of $\epsilon_{i,0}$. $\Gamma_{i,0}$ is characterized by h. H is the cumulative distribution of $\epsilon_{i,0}$. The probability that investor i buys the conventional bond B_0 given alternative bond and $\epsilon_{i,0}$ is:

$$\begin{aligned} Pr_i(B_0|b_{l,0}, \epsilon_{i,0}, \epsilon_{i,l}, \forall l \neq 0) &= Pr(y^*_{i,0} > y^*_{i,l}|\epsilon_{i,0}, \epsilon_{i,l}, \forall l \neq 0), \\ Pr_i(B_0|b_{l,0}, \epsilon_{i,0}, \epsilon_{i,l}, \forall l \neq 0) &= Pr(\Delta\epsilon_{i,l,0} < \Delta y_{i,0,l}, \forall l \neq 0), \\ Pr_i(B_0|b_{l,0}, \epsilon_{i,0}, \forall l \neq 0) &= \prod_{l \neq 0} H(\Delta y_{i,0,l} + \epsilon_{i,0}). \end{aligned} \tag{51}$$

Eq. (50) is a conditional probability. We still need to account for the possible values of $\epsilon_{i,0}$.

$$\begin{aligned} Pr_i(B_0|b_{l,0}, \forall l \neq 0) &= \int_{-\infty}^{+\infty} Pr_i(B_0|b_{l,0}, \epsilon_{i,0}, \forall l \neq 0) h(\epsilon_{i,0}) d\epsilon_{i,0}, \\ Pr_i(B_0|b_{l,0}, \forall l \neq 0) &= \int_{-\infty}^{+\infty} \prod_{l \neq 0} H(\Delta y_{i,0,l} + \epsilon_{i,0}) h(\epsilon_{i,0}) d\epsilon_{i,0}. \end{aligned} \tag{52}$$

Eq. (52) represents the probability that Condition 1 gets satisfied. For example, if $\epsilon_{i,0}$ is represented by an extreme value distribution, we obtain a multinomial logit model (Train, 2009), and $Pr_i(B_0|b_{l,0}, \forall l \neq 0)$ can be written:

$$Pr_i(B_0|b_{l,0}, \forall l \neq 0) = \frac{exp(y_{i,0})}{\sum_{l=0}^{l=L} exp(y_{i,l})}. \tag{53}$$

The Probability of Satisfying Condition 2

Condition 2 is satisfied if investor i chooses bond B_0 over the risk-free bond B_{free}.

The binomial case is a multinomial case with two choices. Therefore, we can deduce that the probability that investor i chooses bond B_0 over the risk-free bond B_{free} is the following:

$$Pr_i(B_0|B_{free}) = \int_{-\infty}^{+\infty} H(\Delta y_{i,0,free} + \epsilon_{i,0})h(\epsilon_{i,0})d\epsilon_{i,0}. \tag{54}$$

If $\epsilon_{i,0}$ is represented by an extreme value distribution, we obtain a multinomial logit model (Train, 2009),

$$Pr_i(B_0|B_{free}) = \frac{exp(y_{i,0})}{exp(y_{i,0}) + exp(y_{i,free})}. \tag{55}$$

The Probability of Satisfying Both Condition 1 and Condition 2

The probability of satisfying both Condition 1 and Condition 2 is the product of both probabilities. This probability is simply the probability that investor i purchases bond B_0:

$$Pr_i(B_0) = Pr_i(B_0|b_{l,0}, \forall l \neq 0)Pr_i(B_0|B_{free}). \tag{56}$$

In the risk-neutral case, we use the budget available to the investor (ξ_i) as their type. If we take into account the type ξ_i, the probability of purchasing bond B_0 belonging to firm l_0 given ξ_i can be written as follows:

$$Pr_i(B_0|\xi_i) = Pr_i(B_0)z(\xi_i), \quad \text{where } 0 < z(\xi_i) < 1. \tag{57}$$

$z(\xi_i)$ is a linear function with the following characteristics:

$$\frac{\partial z(\xi_i)}{\partial \xi_i} > 0. \tag{58}$$

Therefore, the higher the type ξ_i, the higher the probability of purchasing bond B_0 by the investor. The current derivation assumes that the investor will demand a single bond. This could have been true if the investor was purchasing a piece of equipment like a car. However, in the case of bonds, the producer usually purchases a finite amount for two reasons: (i) To have a diversified portfolio to reduce risk, (ii) To minimize transaction costs. In the section below, we demonstrate that the decision to purchase bonds is a discrete/continuous choice problem.

2.4.3 The Discrete/Continuous Choice of the Investor

We follow the procedure used by Hannemann (1984), Chintagunta (1993), and Richards (2000) to investigate the discrete-continuous choice of the investor. These authors considered a consumer that visits a store to purchase a basket of goods. (i) They separated the basket of goods into two groups: one containing only the product category of interest and the other containing all other goods purchased (The composite good). (ii) They assume that there are a certain number of brands in the product category of interest and that the consumer is assumed to purchase at most one of the brands on any store visit. (iii) The consumer purchases several quantities of the brand selected on each store visit. There exists a vector of qualities associated with each brand. The quality of a brand is enhanced by the nonprice marketing variables associated with the brand and also by consumer-specific factors such as brand loyalty. (iv) The consumer is assumed to maximize his utility on every store visit knowing that the attributes of a brand can change on every store visit. Therefore the brand and the chosen quantities can differ on every store visit. (v) The utility maximization is subject to the consumer's budget constraint.

Our problem shares several similarities with the application discussed above: (i) In this essay, the equivalent of the supermarket is the securities market. The investor visits the securities market of bonds, stocks, preferred shares, and ETFs. (ii) investor i chooses bonds as the category of interest. The other securities (stocks, preferred shares, and ETFs) are the composite good. In our case, the composite good can be represented by a risk-free asset, such as a risk-free treasury bond, following the discussion on the derivation of Condition 2. (iii) Within the category of interest, we have different brands of bonds with similar (c,Y, T). We have L heterogeneous firms and, therefore, L different bond brands. (iv) As compared to food products, where the consumer purchases at most one brand, investors tend to purchase a portfolio made of several brands of bonds to diversify the risks in their portfolio and reduce transaction costs. (v) The bonds have several attributes, such as their yields which depend on the probability of default of the bond, and stochastic factors affecting the value of the bond. (vi) In the risk-neutral case, the investor maximizes the expected return of their bond portfolio and the composite good taking into account the transaction costs and his budget constraint. (vii) Like in the food example, the investor needs to select the number of bonds to purchase for each brand of bond in its portfolio.

The above discussions show that our problem can be set as a discrete/continuous choice problem where the investor selects different brands (discrete choice) and then the quantity of each brand to purchase (continuous choice). In the following maximization problem, we assume that there is a finite amount (L) of firms issuing bonds. The investor needs to decide on the number of bonds to purchase from the firm l given the characteristics of the bonds offered by the other L-1 firms and the risk-free bond. $n^d_{i,l}$ is the quantity demanded by investor i of bonds $b_{l,0}$ issued by firm l.

$$\max_{\{n^d_{i,l}\}_{l=0}^{L},\, n^d_{i,free}} \sum_{l=0}^{L} y_{i,l} * b_{l,0} * n^d_{i,l} + y_{i,free} * b_{free} * n^d_{i,free} - \sum_{l=0}^{L} C(n^d_{i,l}) - C(n^d_{i,free}),$$
$$st. \sum_{l=0}^{L} b_l * n^d_{i,l} + b_{free} * n^d_{i,free} \leq \xi_i, \tag{59}$$

with $\Omega_3 = \xi_i - \sum_{l=0}^{L} b_l * n^d_{i,l} + b_{free} * n^d_{i,free}$.

In Lagrangean form, problem (59) can be written as follows:

$$L_{\lambda_3} = \sum_{l=0}^{L} y_{i,l} * b_{l,0} * n^d_{i,l} + y_{i,free} * b_{free} * n^d_{i,free} - \sum_{l=0} C(n^d_{i,l}) - C(n^d_{i,free}) + \lambda_3\left(\sum_{l=0} b_l * n^d_{i,l} + b_{free} * n^d_{i,free} - \xi_i\right), \tag{60}$$

With the following complementarity condition that needs to be met:

$$0 \leq \lambda_3 \perp \Omega_3 \geq 0.$$

The First Order Condition[12] (FOC) of problem (59) is as follows:

$$\begin{aligned}
&\frac{\partial L_{\lambda_3}}{\partial n^d_{i,l}} = y_{i,l} * b_{l,0} - \frac{\partial C}{\partial n^d_{i,l}} - \lambda_3 * b_{l,0} = 0, \\
&\frac{\partial L_{\lambda_3}}{\partial n^d_{i,free}} = y_{i,free} * b_{free} - \frac{\partial C}{\partial n^D_{i,free}} - \lambda_3 * b_{free} = 0, \\
&\lambda_3 \geq 0, \\
&\Omega_3 \geq 0, \\
&\lambda_3 * (\Omega_3) = 0.
\end{aligned} \tag{61}$$

The cost function is such that the higher the quantity of bond purchased, the lower the transaction cost. The cost function has a convex shape, so the negative of the cost function has a concave shape.

$$\begin{aligned}
&\frac{\partial C}{\partial n^d_{i,l}} < 0. \\
&\frac{\partial^2 C}{\partial {n^d_{i,l}}^2} > 0.
\end{aligned} \tag{62}$$

When C is specified as above, we are guaranteed a unique set of optimal solutions because the problem (59) becomes a convex optimization problem. For firm l_0, solving problem (59) generates the optimal quantity ($n^{d,*}_{i,l_0}$) chosen by investor i for bond B_0:

$$n^{d,*}_{i,l_0} = n^{d,*}_{i,l_0}(b_0 | b_{l,0}, b_{free}, E_i(P_t), E_i(R_t), E_i(\tau_t), \Gamma_C, c, \Upsilon, T, \alpha, \omega, T_0, \xi_i, \Gamma_F) \ \forall l = 0. \tag{63}$$

Where Γ_C represents the parameters of the transaction cost function.

2.4.4 The Aggregate Demand for the Conventional Bond Issued by a Given Firm

Suppose the market is made of risk-neutral investors of type ξ_i. Let I be the population of investors. n^*_{l,l_0} is the aggregate demand of bonds averaging over the type of investors. Let g_1 be the population distribution of ξ_i:

$$n^{d,*}_{l,l_0} = \int Pr(B_0|\xi_i) n^{d,*}_{i,l_0}(b_0 | b_{l,0}, b_{free}, E_i(P_t), E_i(R_t), E_i(\tau_t), \ldots \tag{64}$$
$$\Gamma_C, c, \Upsilon, T, \alpha, \omega, T_0, \xi_i) g_1(\xi_i | \Gamma_{\xi_i}) d(\xi_i) \text{ with } l = 0$$

$n^{d,*}_{l,l_0}$ represents the market demand for bond B_0 emitted by firm l_0. Γ_{ξ_i} represents the parameters of the distribution of ξ_i. In equation (63), we integrate out over the population distribution of ξ_i to compute the demand for conventional bonds (B_0) over all the alternatives by randomly selected investors.

[12]This optimization problem can be solved in two steps. First, we check if the bond is better than others, then choose how many to buy

$$n^{d,*}_{l,l_0} = n^{d,*}_{l,l_0}\Big(b_0 | b_{l,0}, b_{free}, E_i(P_t), E_i(R_t), E_i(\tau_t), \delta_i, \Gamma_{i,0}, \Gamma_C, \ldots, \Gamma_{\xi_i}, c, \Upsilon, T, \alpha, \omega, T_0, \Gamma_F, \Gamma_I\Big) \quad \text{with } l \neq 0 \tag{65}$$

Eq. 65 shows that the quantity of bonds B_0 demanded by investors depends on the price offered on other bonds in the market. If we had not modeled the demand as a discrete/continuous choice problem, the demand for bond B_0 would only depend on b_0, which is unrealistic. Moreover, the discrete/continuous choice specification allows us to consider the transaction costs at the time of purchase.

2.4.5 The Determination of the Equilibrium Market Price

We consider a setting with few single product firms (only one brand of bond emitted per firm). Here, the product is the bond issued by the firm. The bonds are differentiated by the company names (the brands) and the intrinsic bonds attributes (coupon, maturity, face value). The existence and uniqueness of price equilibrium are important from both a theoretical and an empirical viewpoint. Among the theoretical studies dealing with this problem, we mention Caplin and Nalebuff (1991), Anderson, de Palma, and Thisse (1992), and Peitz (2000). All these studies assume that firms produce one product. The first study analyzes the existence and uniqueness of price equilibrium for several models, including random coefficient discrete choice models. These authors allow for fairly general distributions of the random coefficients. They establish the existence of price equilibrium for, among others, mixed logit models and the existence and uniqueness of price equilibrium for the standard logit, both for the linear income-price difference specification and the log income-price difference specification. Under the intuitive assumption that the purchase probabilities of bonds are decreasing in own prices and increasing in the prices of rival products, we are guaranteed that the pricing game has a unique equilibrium for the standard logit with linear income-price difference specification. In this paper, we assume a standard logit demand with a linear income-price difference because we are guaranteed a unique equilibrium under this specification (Feenstra and Levinsohn (1995), BLP, Nevo (2000) and Nevo (2001).

Hanson and Martin (1996) are the first to show that the logit profit function is not concave in price. While the objective function is not concave in the price vector, Dong et al. (2009) and Song and Xue (2007) show that the standard MNL model is concave with respect to the market share vector, which is a one-to-one transformation of the price vector. The concavity of the objective function allows for deriving a single-dimension search solution for the optimal prices and market shares.

We consider a setting where firm l_0 decides on its price (b_0) knowing the price offered by other firms in the market for bonds similar to bond B_0. The price offered by firm l_0 takes into account the price offered by other firms in the market. At equilibrium, we assume the price b_0 adjusts and is known. The optimal supply of conventional bonds from firm l_0 (eq. 33) is equal to the optimal demand for conventional bonds of firm l_0 (eq 2.64):

$$n^{s,*}_{l_0}\left(b_0 | c, Y, E_l(P_t), E_l(R_t), E_l(\tau_t), \alpha, \delta_{l_0}, \omega, T, T_0, \Gamma_F\right) = n^{d,*}_{l,l_0}\left(b_0 | b_{l,0}, b_{free}, E_i(P_t), E_i(R_t), E_i(\tau_t), \delta_i, \Gamma_{i,0}, \Gamma_C, \Gamma_{\xi_i}, c, Y, T, \alpha, \omega, T_0, \Gamma_F\right), \tag{66}$$

with $l \in \{1, \ldots, L-1\}$.

From equation (66), we can obtain, under risk neutrality, the equilibrium price of the conventional bond (b^*_0). The equilibrium price is the only price where the desires of investors and the desires of producers agree—that is, where the amount of bonds that investors want to buy ($n^{d,*}_{l,l_0}$) is equal to the amount firm l wants to sell ($n^{s,*}_{l_0}$). This mutually desired amount is called the equilibrium quantity.

We developed a pricing theory not based on the unrealistic assumptions of the CAPM and Black-Scholes models (constant volatility, constant interest rate for borrowing and lending, Brownian motion). More importantly, unlike the empirical papers, our theory approach is as general as possible and is not sample-dependent, location-dependent, or time-dependent. The Black-Scholes model is solely based on firm characteristics and macroeconomic market conditions, and the CAPM is based on investor/bond portfolio characteristics; our equilibrium price considers both market forces.

Several hypotheses can be generated regarding the factors affecting the equilibrium price at issuance of a conventional bond:

Hypothesis I (Under Risk Neutrality): On the supply side, the price of a conventional bond is affected by the characteristics of the bond (c,Y, T), subjective expectations of the firm towards stochastic variables (input price, output price, and weather), the discount rate used by the firm reflecting the perception of the firm regarding the macroeconomic risk, the characteristics of the technology purchased with the bond proceeds (Time of obsolescence, deterioration rate), the banking system interest rate, and the production technology specification. The hypothetical sign of the effect of those variables on the supply of conventional bonds can be found in Table 1.

Hypothesis II (Under Risk Neutrality): On the demand side, the price of a conventional bond is affected by the characteristics of the bond (c,Y, T), the price offered by other firms in the market for a bond of similar characteristics, the price of the risk-free asset, the discount rate used by investors reflecting their perceptions of the macroeconomic risk, technological factors (time of obsolescence, deterioration rate) and subjective expectations of investors towards stochastic variables (input price, output price, and weather) affecting the probability of pecuniary default of the firm, the banking system interest rate, the transaction cost in the bond market, and the characteristics of the investors determined by their income levels in the risk-neutral case. The hypothetical sign of the effect of those variables on demand for conventional bonds can be found in 1.

2.5 The Theory of Conventional Bond Supply Under Risk Aversion

Although we assume in this essay that the producer is risk-neutral. It is worth discussing briefly how the issuer supply could be affected if the firm were risk-averse. William Schworm (1980), Appelbaum, and Harris (1978) indicate that investment behavior is sensitive to the risk aversion of the firm. This idea is backed by Stigltz (1969) and Cohn et al. (1975), who show that capital investment is reduced with the firm's degree of absolute risk aversion. We deduce that the quantity supplied of bonds by the firm in the risk-averse case will be lower than in the risk-neutral case. Let $\beta_l \in [0, 1]$ be the risk-aversion of firm l:

$$\frac{\partial n_l^{s,*}(\beta_l)}{\partial \beta_l} < 0. \tag{67}$$

Where $n_l^{s,*}(\beta_l)$ is the optimal amount of bond supplied by the firm under risk aversion.

2.6 The Theory of Investor Demand under Risk Aversion

Under risk aversion, the investor has two characteristics or types: their level of income ξ_i and their level of risk aversion β_i. The investor has a direct utility function $U_{i,l}$ such that if bond B_0 is selected then:

$$U_{i,0}(B_0|\xi_i, \beta_i) > U_{i,l}(b_{l,0}|\xi_i, \beta_i), \ \forall\ l \neq 0. \tag{68}$$

Eq. (68) implies that the investor selects bond B_0 because it is the choice that provides him with the highest utility given ξ_i, β_i. Let $V_{i,l}$ be the indirect utility function of investor i for bond $b_{l,0}$. The computation of yield stays the same as in the risk-neutral case.

Optimality conditions require the following equality between the direct and the indirect utility functions of investor i:

$$V_{i,l}(y_{i,l}) \equiv U_{i,l}(b_{l,0}|\xi_i, \beta_i), \ \forall\ d \neq 0. \tag{69}$$

We specify $V_{i,l}$ such that $\frac{\partial V_{i,l}}{\partial y_{i,l}} > 0$. The signs of the second partial derivatives with respect to yield $\frac{\partial^2 V_{i,l}}{\partial y_{i,l}^2}$ depend on β_i. The convexity of $V_{i,l}$ varies with β_i. We assume the independence of β_i over the population of investors. For $\beta_i = 0$, the investor i is risk-neutral, so the second partial derivatives are equal to 0. For $\beta_i > 0$, the investor i is risk averse, so his indirect utility function is concave, meaning that the second partial derivatives with respect to yield are negative. As to ξ_i, we define it such that $\frac{\partial V_{i,l}}{\partial \xi_i} > 0$, ie. investors with higher incomes have higher utilities. However, $\frac{\partial^2 V_{i,l}}{\partial \xi_i^2} < 0$.

Following Train's (2009) multinomial choice approach, $V_{i,l}$ is an observable indirect utility to the researcher and is defined with eq. (69). $\epsilon_{V_{i,l}}$ is the random indirect utility of yield; the researcher does not see this but knows its distribution. Therefore, $V_{i,l}^*$, the actual indirect utility of the investor, is unobservable to the researcher because of $\epsilon_{V_{i,l}}$. investor i receives $V^*_{i,l}$ as indirect utility of yield, where:

$$V^*_{i,l} = V_{i,l} + \epsilon_{V_{i,l}}. \tag{70}$$

The maximization problem involves comparing actual yields. investor i chooses bond B_0 over L-1 alternative bonds if two conditions are met:

$$\begin{aligned} &\textbf{Condition 1} \quad V^*_{i,0} > V^*_{i,l}, \forall l \neq 0, \\ &\textbf{Condition 2} \quad V^*_{i,0} > V^*_{i,free}. \end{aligned} \tag{71}$$

The second condition can be derived from the first condition using the relationship between the portfolio of alternative bonds $\overline{B}$ and the risk-free bond (eq. 71). Therefore, the multinomial choice of the investor to choose B_0 relative to alternative corporate bonds (Condition 1) implies that the investor must also prefer to choose B_0 compared to a risk-free bond (Condition 2). Condition 1 implies Condition 2, but the reverse is not always true.

The Probability of Satisfying Condition 1

Let $\Delta\epsilon_{V_{i,l,0}} = \epsilon_{V_{i,l}} - \epsilon_{V_{i,0}}$ and $\Delta V_{i,l} = V_{i,l} - V_{i,0}$. Condition 1 can be equivalently written as follows:

$$\begin{aligned} &\epsilon_{V_{i,l}} - \epsilon_{V_{i,0}} < V_{i,0} - V_{i,l}, \forall l \neq 0, \\ &\Delta\epsilon_{V_{i,l,0}} < \Delta V_{i,0,l} \forall l \neq 0. \end{aligned} \tag{72}$$

Because of $\epsilon_{V_{i,0}}$'s and $\epsilon_{V_{i,l}}$'s, the decision of investor i to choose B_0 over all other firm bonds $b_{l,0}$ ($l \neq 0$) is a stochastic choice, from the researcher's point of view. Specific assumptions about the distribution of $\epsilon_{V_{i,0}}$'s will determine investor i's choice probabilities. Let h be the distribution of $\epsilon_{V_{i,0}}$. $\Gamma_{i,0}$ is characterized by h. H is the cumulative distribution of $\epsilon_{V_{i,0}}$. The probability that investor i buys the conventional bond B_0 given alternative bonds and $\epsilon_{V_{i,0}}$ is:

$$\begin{aligned} &Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \epsilon_{V_{i,0}}, \epsilon_{V_{i,l}}, \forall l \neq 0) = Pr(V^*_{i,0} > V^*_{i,l}|\epsilon_{V_{i,0}}, \epsilon_{V_{i,l}}, \forall l \neq 0), \\ &Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \epsilon_{V_{i,0}}, \epsilon_{V_{i,l}}, \forall l \neq 0) = Pr(\Delta\epsilon_{V_{i,l,0}} < \Delta V_{i,0,l}, \forall l \neq 0), \\ &Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \epsilon_{V_{i,0}}, \forall l \neq 0) = \prod_{l \neq 0} H(\Delta V_{i,0,l} + \epsilon_{V_{i,0}}). \end{aligned} \tag{73}$$

Eq. (73) is a conditional probability. We still need to account for the possible values of $\epsilon_{V_{i,0}}$.

$$\begin{aligned} &Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \forall l \neq 0) = \int_{-\infty}^{+\infty} Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \epsilon_{V_{i,0}}, \forall l \neq 0) h(\epsilon_{V_{i,0}}) d\epsilon_{V_{i,0}}, \\ &Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \forall l \neq 0) = \int_{-\infty}^{+\infty} \prod_{l \neq 0} H(\Delta V_{i,0,l} + \epsilon_{V_{i,0}}) h(\epsilon_{V_{i,0}}) d\epsilon_{V_{i,0}}. \end{aligned} \tag{74}$$

Eq. (74) represents the probability that Condition 1 gets satisfied. For example, if $\epsilon_{V_{i,0}}$ is represented by an extreme value distribution, we obtain a multinomial logit model (Train, 2009), and $Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \forall l \neq 0)$ can be written:

$$Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \forall l \neq 0) = \frac{exp(V_{i,0})}{\sum_{d=0}^{d=} exp(V_{i,l})}. \quad (75)$$

The Probability of Satisfying Condition 2

Condition 2 is satisfied if investor i chooses bond B_0 over the risk-free bond B_{free}. The binomial case is a multinomial case with two choices. Therefore, we can deduce that the probability that investor i chooses bond B_0 over the risk-free bond B_{free} is the following:

$$Pr_i(B_0|B_{free}, \xi_i, \beta_i) = \int_{-\infty}^{+\infty} H(\Delta V_{i,0,free} + \epsilon_{V_{i,0}})h(\epsilon_{V_{i,0}})d\epsilon_{V_{i,0}}. \quad (76)$$

If $\epsilon_{V_{i,0}}$ is represented by an extreme value distribution, we obtain a multinomial logit model (Train, 2009),

$$Pr_i(B_0|B_{free}, \xi_i, \beta_i) = \frac{exp(V_{i,0})}{exp(V_{i,0}) + exp(V_{i,free})}. \quad (77)$$

The Probability of Satisfying Conditions 1 and 2

The probability of satisfying both Condition 1 and Condition 2 is the product of both probabilities. This probability is simply the probability that investor i purchases bond B_0 given his characteristics:

$$Pr_i(B_0|\xi_i, \beta_i) = Pr_i(B_0|b_{l,0}, \xi_i, \beta_i, \forall l \neq 0)Pr_i(B_0|B_{free}, \xi_i, \beta_i). \quad (78)$$

As in the risk-neutral case, we model the decision to purchase bonds as a discrete/continuous choice problem.

2.6.1 The Discrete/Continuous Choice of the Investor

We follow the procedure used by Hannemann (1984), Chintagunta (1993), and Richards (2000) to investigate the discrete-continuous choice of the investor. We assume that there is a finite amount (L) of firms issuing bonds. The investor needs to decide on the number of bonds to purchase from the firm l given the characteristics of the bonds offered by the other L-1 firms and the risk-free bond. $n^d_{i,l}$ is the quantity demanded by investor i of bonds $b_{l,0}$ issued by firm l. The investor selects the quantities that maximize their total net utility subject to its budget constraint.

$$\max_{\{n^d_{i,l}\}_{l=0}^{L}, n^d_{i,free}} \sum_{l=0}^{L} V_{i,l} * n^d_{i,l} + V_{i,free} * n^d_{i,free} - \sum_{l=0}^{L} C(n^d_{i,l}) - C(n^d_{i,free}),$$
$$st. \sum_{l=0}^{L} b_l * n^d_{i,l} + b_{free} * n^d_{i,free} \leq \xi_i, \quad (79)$$

with $\Omega_4 = \xi_i - \sum_{l=0}^{L} b_l * n^d_{i,l} + b_{free} * n^d_{i,free}$.

C is the transaction cost of purchasing a given number of bonds $b_{l,0}$ issued by firm l.

The first and second derivatives of C are defined similarly as in the risk-neutral case (eq. 62). In Lagrangean form, problem (79) can be written as follows:

$$L_{\lambda_4} = \sum_{l=0}^{L} V_{i,l} * n^d_{i,l} + V_{i,free} * n^d_{i,free} - \sum_{l=0}^{L} C(n^d_{i,l}) - C(n^d_{i,free}) + \lambda_4 \left(\sum_{l=0}^{L} b_l * n^d_{i,l} + b_{free} * n^d_{i,free} - \xi_i \right). \tag{80}$$

With the following complementarity condition that needs to be met:

$$0 \leq \lambda_4 \perp \Omega_4 \geq 0.$$

The First Order Condition (FOC) of problem (79) is as follows:

$$\begin{aligned}
&\frac{\partial L_{\lambda_4}}{n^d_{i,l}} = V_{i,l} - \frac{\partial C}{\partial n^d_{i,l}} - \lambda_4 * b_{l,0} = 0,\\
&\frac{\partial L_{\lambda_4}}{n^d_{i,free}} = V_{i,free} - \frac{\partial C}{\partial n^D_{free,i}} - \lambda_4 * b_{free} = 0,\\
&\lambda_4 \geq 0,\\
&\Omega_4 \geq 0,\\
&\lambda_4 * (\Omega_4) = 0.
\end{aligned} \tag{81}$$

Solving problem (79) generates the optimal quantity ($n^{d,*}_{i,l_0}$) chosen by investor i for bond B_0:

$$n^{d,*}_{i,l_0} = n^{d,*}_{i,l_0}(b_0 | c, \Upsilon, T, E_i(P_t), E_i(R_t), E_i(\tau_t), \alpha, \omega, T_0, \xi_i, \delta_i, \theta_i, b_{l,0}, b_{free}, \Gamma_C, \Gamma_V) \forall l = 0. \tag{82}$$

Where Γ_C represents the parameters of the transaction cost function. Γ_V represents the parameters defining the utility function.

2.6.2 The Aggregate Demand for the Conventional Bond Issued by a Given firm

Suppose the market is made of risk-averse investors of type ξ_i. $n^{d,*}_{l,l}$ is the aggregate demand of bonds averaging over the type of the investors. g_1 is the population distribution of ξ_i, and g_2 is the population distribution of θ_i:

$$n^{d,*}_{l,l_0} = \int\int Pr(B_0 | \xi_i, \theta_i) n^{D,*}_{l,i} * g_1(\xi_i | \Gamma_{\xi}) g_2(\theta_i | \Gamma_{\theta}) \, d(\xi_i) \, d(\theta_i). \tag{83}$$

$n^{d,*}_{l,l_0}$ represents the market demand for bond B_0 emitted by firm l. Γ_{ξ_i} represents the parameters of the distribution of ξ_i. Γ_{θ_i} represents the parameters of the distribution of θ_i in equation (83); we integrate out over the population distributions of ξ_i and θ_i to compute the demand for conventional bonds (B_0) over all the alternatives by randomly selected investors.

$$n^{d,*}_{l,l_0} = n^{d,*}_{l,l_0}\left(b_0 | c, \Upsilon, T, E_i(P_t), E_i(R_t), E_i(\tau_t), \alpha, \omega, T_0, b_{l,0}, b_{free}, \delta_i, \Gamma_{i,0}, \Gamma_C, \Gamma_V, \Gamma_{\xi_l}, \Gamma_{\beta_i} \right) \text{ with } l = 0. \tag{84}$$

Γ_{β_i} represents the parameters defining the distribution of the risk-aversion coefficient within the population.

2.6.3 The Determination of the Equilibrium Market Price

The existence and uniqueness of a price equilibrium have already been discussed in the risk-neutral case. When the demand is specified as a multinomial logit, we are guaranteed that the price equilibrium exists with a few single-product firms. Moreover, when the demand has a linear income-price difference specification, we are guaranteed that the price equilibrium is unique (Caplin and Nalebuff (1991), Anderson, de Palma and Thisse (1992), Peitz (2000), Feenstra and Levinsohn (1995), BLP, Nevo (2000) and Nevo (2001), Hanson and Martin (1996). Like in the risk-neutral case, we assume that the firm decides on the number of bonds to issue or produce, and the price is determined after other firms in the market have revealed their prices.

At equilibrium, we assume the price b_0 adjusts to being known. The optimal supply of conventional bonds from firm l_0 is equal to the optimal demand for conventional bonds of firm l_0:

$$\begin{aligned} n^{s,*}_{l_0,i} &= n^{s,*}_{l_0,i}(b_0 | c, \Upsilon, E_l(P_t), E_l(R_t), E_l(\tau_t), \alpha, \delta_{l_0}, \omega, T, T_0), \\ n^{d,*}_{l,l_0} &= n^{d,*}_{l,l_0}\left(b_0 | c, \Upsilon, T, E_i(P_t), E_i(R_t), E_i(\tau_t), \alpha, \omega, T_0, b_{l,0}, b_{free}, \delta_i, \ldots \right. \\ &\quad \left. \Gamma_{i,0}, \Gamma_C, \Gamma_{\xi_l}, \Gamma_V, \Gamma_{\beta_i} \right) \text{ with } l \in \{1, \ldots, l-1\}. \end{aligned} \tag{85}$$

We note that the financial literature cautions that there are limited conditions under which an equilibrium price exists under risk aversion (Heidhues, 2008[13]). For example, Steven Gjerstad (2005) shows that when investors have Constant Relative Risk Aversion (CRRA) expected utility functions, the equilibrium bias diminishes as the coefficient of relative risk aversion increases, and the bias disappears when the coefficient of relative risk aversion is 1. At that point, the equilibrium price equals the mean belief for any distribution of beliefs. If the coefficient of relative risk aversion exceeds 1, the direction of the bias reverses.

Eq. (85) allows us to generate a third hypothesis on the demand side of the conventional bond.

Hypothesis III (Under Risk Aversion): On the demand side, compared to the risk-neutral case, the price of the conventional bond is additionally affected by the parameters defining the utility of the investors, and the distribution of the risk aversion coefficient within the population. The hypothetical sign of the effect of those variables on demand for conventional bonds can be found in Table 1 (Appendix).

[13]Competition and Price Variation When Consumers Are Loss Averse

3 Green Bond Pricing Theory

3.1 Assumptions

Several differences exist between the conventional and the green bond pricing theory: (i) The conventional bond issuer does not consider the public outputs they emit when evaluating their optimal contemporaneous and long-run profits. In the green bond case, we consider that $S^*_{l,t}$ represents the short-term optimal amount of carbon emissions of the firm during period ($t > 0$). The firm tracks this quantity to evaluate the firm green performance at the bond maturity.

$$S^*_{l,t} \equiv S^*_{l,t}\left(\vartheta_{l,t}(k_{l,0}),\ E_l(P_t),\ E_l(R_t),\ E_l(\tau_t)\right).$$

(ii) Let $\hat{S}^*_{l,T}$ be the optimal total amount of carbon emissions of the firm during the period of maturity of the bond (Figure 1).

$$\hat{S}^*_{l,T} \equiv \sum_{t=0}^{t=T} S^*_{l,t}. \tag{86}$$

$$\hat{S}^*_{l,T} \equiv \hat{S}^*_{l,T}\left(\{\vartheta_{l,t}(k_{l,0}),\ E_l(\tau_t),\ E_l(P_t),\ E_l(R_t)\}_{t=0}^{t=T}\right).$$

Let $\hat{S}^*_{l,0}$ represents the optimal total amount of carbon emissions of the firm in the period of length T preceding the time of issuance (t=0). We assume that $\hat{S}^*_{l,0}$ is known at the time of issuance and is used by the green bond issuer as a reference to which $\hat{S}^*_{l,T}$ will be compared to in the future.

$$\hat{S}^*_{l,0} = \sum_{t=-T}^{t=0} S^*_{l,t}. \tag{87}$$

(iii) At the time of issuance (t=0), the firm promises to bondholders that $\hat{S}^*_{l,T}$ will

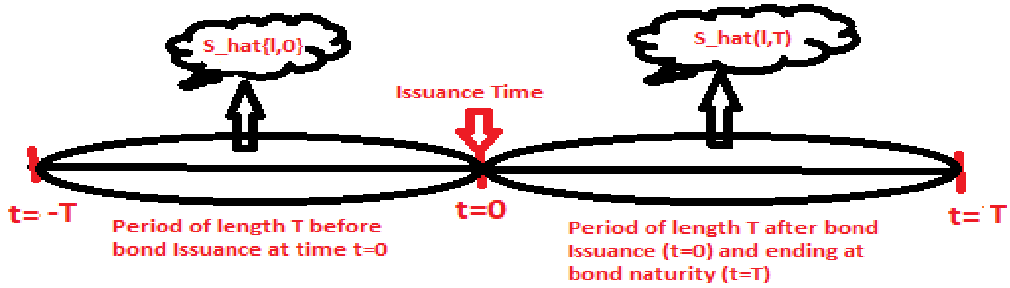

Figure 1: Illustration of Pre-Issuance Total Level of Carbon Emissions ($\hat{S}^*_{l,0}$) and Post-Issuance Total Level of Carbon Emissions ($\hat{S}^*_{l,T}$)

be at most equal to $\tilde{S}_{l,T}$, the target level of total carbon emissions post-issuance. The firm sets $\tilde{S}_{l,T}$ as a percentage ρ of $\hat{S}^*_{l,0}$. $\rho \in (0,1)$ because the target of the firm must be smaller than the pre-issuance level ($\hat{S}^*_{l,0}$). However, the firm understands that $\hat{S}^*_{l,T}$ is stochastic, so the promised target may not be met. $\hat{S}^*_{l,T}$ is stochastic because the green technology used by the firm for self-protection depends on the weather. Moreover, input price and output price fluctuations during the bond maturity may lead the firm to emit more carbon emissions than expected to stay profitable. Let $J_{l,T}$ be the error in the environmental impact of the bond $b_{l,0}$ defined as follows:

$$
\begin{aligned}
&J_{l,T} \equiv \hat{S}^*_{l,T} - \tilde{S}_{l,T},\\
&J_{l,T} \equiv \hat{S}^*_{l,T} - \rho * \hat{S}^*_{l,0},\\
&J_{l,T} \equiv J_{l,T}\left[\{\vartheta_{l,t}(k_{l,0}), E_l(\tau_t), E_l(P_t), E_l(R_t)\}_{t=0}^{t=T}, \rho, \hat{S}^*_{l,0}\right],\\
&\text{If } J_{l,T} > 0 \implies \text{the firm has committed green default},\\
&\text{If } J_{l,T} \leq 0 \implies \text{the firm has fulfilled its green promise.}
\end{aligned}
\tag{88}
$$

(iv) Compared to a conventional bond, a green bond is defined by its maturity level (T), its coupon rate (c), its face value (Y), and the environmental outcome target at maturity $\tilde{S}_{l,T}$.

(v) Compared to conventional bonds, the issuance cost of green bonds is not negligible ($IC \neq 0$). IC embodies the additional expenditures related to a green bond issuance (extra investment costs, separate accounting, additional monitoring, and reporting) (Agliardi et al., 2021). IC depends on the number of bonds issued. The higher the size of the investment, the more effort and scrutiny are needed by the firm to monitor the environmental performance of the bonds.

(vi) The market is made of investors having different prosocial attitudes (η_i), levels of risk aversion (β_i), and different income levels (ξ_i). η_i and β_i are defined as scalar values. The hedonic investor ($\eta_i = 0$) is only motivated by the yield from the bond $b_{l,0}$ ($y_{i,l}$). Whereas a prosocial investor ($\eta_i = 1$) values the yield ($y_{i,l}$) but also the environmental impact of the bond ($J_{l,T}$). The risk preference and the investor's prosocial attitude determine the investor's utility function. To specify this, we focus on indirect utility as specified in 2.68.

3.2 The Theory of the Green Bond Issuer Supply

The short-term profit of the firm under a green bond issuance is derived similarly to the short-term profit under a conventional bond issuance (eq. 14). The only difference is that the green bond issuer controls the amount of carbon emitted in the short run. We then define the long-run restricted profit function of the firm issuing a green bond by modifying the firm issuing the conventional bond (eq. 23).

$$\begin{aligned}
\Pi^{lr,*}_{l,t} &= n_l b_0 + \Pi^{sr,*}_{l,t} - IC \text{ for } t = 0,\\
\Pi^{lr,*}_{l,t} &= \Pi^{sr,*}_{l,t} - n_l c\Upsilon \text{ for } 1 < t < T,\\
\Pi^{lr,*}_{l,t} &= \Pi^{sr,*}_{l,t} - n_l c\Upsilon - n_l\Upsilon \text{ for } t = T,\\
\Pi^{lr,*}_{l,t} &= \Pi^{sr,*}_{l,t} \text{ for } T < t \leq T_0.
\end{aligned} \tag{89}$$

Eq. 89 shows that when the firm emits a bond, it receives $n_l b_0$ initially but needs to pay $n_l c\Upsilon$ over the next T periods, as well as the cost of issuance (IC). We can derive the net present value of the long-term profit that will be obtained by the firm over the life of the investment as follows:

$$\begin{aligned}
\Pi^{lr,*}_{l} &= NPV\left(\Pi^{lr,*}_{l,t}\right),\\
&= n_l b_{l,0} - IC\left(n_l\right) + \sum_{t=0}^{T_0} \delta_l^t \Pi^{sr,*}_{l,t} - \sum_{t=1}^{T} \delta_l^t \left(n_l c\Upsilon\right) - \delta_l^T \left(n_l \Upsilon\right).
\end{aligned} \tag{90}$$

Following the same optimization process described in the conventional bond case, we derive the optimal level of investment $k^*_{l,0}$ as dependent on the following exogenous variables:

$$k^*_{l,0} = k^*_{l,0}(b_l, c, \Upsilon, E_l(P_t), E_l(R_t), E_l(\tau_t), \alpha, \delta_l, \omega, T, T_0, \Gamma_F, \Gamma_{IC}, \rho, \hat{S}^*_{l,0}). \tag{91}$$

Γ_{IC} are respectively the parameters of the cost of issuance function. IC's first and second derivatives have been discussed in the conventional bond case.

Knowing $k^*_{l,0}$, the optimal level of bonds that will be supplied by firm l can be obtained as follows:

$$\begin{aligned}
n^{s,*}_l &= \frac{k^*_{l,0}}{b_{l,0}},\\
n^{s,*}_l &= n^{s,*}_l(b_l, c, \Upsilon, E_l(P_t), E_l(R_t), E_l(\tau_t), \alpha, \delta_l, \omega, T, T_0, \Gamma_{IC}, \Gamma_F, \rho, \hat{S}^*_{l,0}).
\end{aligned} \tag{92}$$

Equation (92) is written in the reduced form regarding exogenous parameters. Compared to the conventional bond, the optimal amount of green bond supplied by the firm depends on the green effect generated by the bond, as well as the issuance cost function.

3.3 The Theory of Green Bond Demand under Risk Neutrality and Hedonic Preferences

We similarly derive the pecuniary return and the demand for green bonds under risk neutrality. We derived the financial return and the demand for conventional bonds under risk neutrality (see sec. 3.2.4).

3.4 The Theory of Green Bond Demand under Risk Aversion and Hedonic Preferences

We derive the demand for green bonds under risk aversion similarly to the demand for conventional bonds under risk aversion (see sec. 3.2.6).

3.5 The Theory of Green Bond Demand under Risk Aversion with both Hedonistic and Prosocial Preferences

3.5.1 The Individual Investor Choice to Buy a Green Bond vs. Alternative Bonds under Risk Aversion

We add in this section the investor's prosocial behavior (η_i). The investor has a direct utility function $U_{i,l}$ such that if the green bond B_0 is selected then:

$$U_{i,l}(B_0|\xi_i, \beta_i, \eta_i) > U_{i,l}(b_{l,0}|\xi_i, \beta_i, \eta_i), \ \forall\, l \neq 0. \quad (93)$$

Eq. (93) implies that the investor selects bond B_0 because it is the choice that provides him with the highest utility. Let $V_{i,l}$ be the indirect utility function of investor i for bond $b_{l,0}$. Since a green bond is represented by its expected yield ($y_{i,l}$), and its expected environmental impact $J_{i,l,T}$.

If B_0 is selected by investor i, this implies that the indirect utility of B_0 is higher than the alternative bonds' indirect utility. The computation of expected yield stays the same as in the conventional bond case (section 3.2.4).

Optimality conditions imply the following equivalence between the direct and the indirect utility functions of investor i:

$$V_{i,l}(y_{i,l}, J_{i,l,T}|\xi_i, \eta_i, \beta_i) \equiv U_{i,l}(b_{l,0}|\xi_i, \beta_i, \eta_i), \ \forall\, l \neq 0. \quad (94)$$

Green bonds attract prosocial investors because of the subjective expected environmental impact of the bond at maturity as held by investor i ($J_{i,l,T}$). $J_{i,l,T}$ is based on the subjective expectations of investor i regarding the input prices, the output prices, and the weather. The investor estimates the issuer's optimal amount of carbon emissions using the procedure described in the green bond supply section, given their expectations about the climate and input and output prices.

$$J_{i,l,T} \equiv J_{i,l,T}\left[\{\vartheta_{l,t}(k^*_{l,0}), E_i(\tau_t), E_i(P_t), E_i(R_t)\}_{t=0}^{t=T}\rho, \hat{S}^*_{l,0}\right]. \quad (95)$$

Where β_i represents the risk aversion level of the investor, η_i represents the prosocial attitude of the investor, and ξ_i represents the income level of the investor. We specify $V_{i,l}$ such that $\frac{\partial V_{i,l}}{\partial y_{i,l}} > 0$. The signs of the second partial derivatives with respect to yield $\frac{\partial^2 V_{i,l}}{\partial y_{i,l}^2}$ depend on β_i. The convexity of $V_{i,l}$ varies with β_i. We assume the independence of β_i over the population of investors. For $\beta_i = 0$, the investor i is risk-neutral, so the second partial derivatives are equal to 0. For $\beta_i > 0$, the investor i is risk averse, so his indirect utility function is concave, meaning that the second partial derivatives are negative. For the prosocial investor, the higher the size of the environmental impact, the higher the indirect utility of the investor. $\frac{\partial V_{i,l}}{\partial J_{i,l,T}} > 0$. The prosocial investor has a concave indirect utility with respect to $J_{i,l,T}$, i.e., $\frac{\partial^2 V_{i,l}}{\partial J^2_{i,l,T}} < 0$ to allow for diminishing marginal indirect utility with respect to $J_{i,l,T}$.

$V_{i,l}$ is observable indirect utility to the researcher and defined with eq.(94). $\epsilon_{V_{i,l}}$ is the random indirect utility of yield; the researcher does not see this but knows its distribution. Therefore, $V^*_{i,l}$, the actual indirect utility of the bond observed by the investor but unobservable to the researcher because of $\epsilon_{V_{i,l}}$

$$V^*_{i,l} = V_{i,l} + \epsilon_{V_{i,l}}. \tag{96}$$

The maximization problem involves comparing actual yields. investor i chooses bond B_0 over L-1 alternative bonds if two conditions are met:

$$\begin{aligned} &\textbf{Condition 1} \quad V^*_{i,0} > V^*_{i,l},\ \forall l \neq 0, \\ &\textbf{Condition 2} \quad V^*_{i,0} > V^*_{i,free}. \end{aligned} \tag{97}$$

The Probability of Satisfying Condition 1

Under risk aversion, the probability of purchasing green bond B_0 over L-1 alternative bonds is derived in a similar fashion that we derived the probability of purchasing conventional bond B_0 over L-1 alternative bonds. The only difference between the two probabilities is that in the green bond case, the probability is also conditioned by the investor's prosocial attitude (η_i).

$$\begin{aligned} Pr_i(B_0|b_{l,0}, \xi_i, \eta_i, \beta_i, \forall l \neq 0) &= \int_{-\infty}^{+\infty} Pr_i(B_0|b_{l,0}, \xi_i, \eta_i, \beta_i, \epsilon_{V_{i,0}}, \forall l \neq 0) h(\epsilon_{V_{i,0}}) d\epsilon_{V_{i,0}}, \\ Pr_i(B_0|b_{l,0}, \xi_i, \eta_i, \beta_i, \forall l \neq 0) &= \int_{-\infty}^{+\infty} \prod_{l \neq 0} H(\Delta V_{i,0,l} + \epsilon_{V_{i,0}}) h(\epsilon_{V_{i,0}}) d\epsilon_{V_{i,0}}. \end{aligned} \tag{98}$$

The Probability of Satisfying Condition 2

Similarly, the probability that Condition 2 is satisfied, i.e., investor i chooses green bond B_0 over the risk-free bond B_{free}.

$$Pr_i(B_0|B_{free}, \xi_i, \eta_i, \beta_i) = \int_{-\infty}^{+\infty} H(\Delta V_{i,0,free} + \epsilon_{V_{i,0}}) h(\epsilon_{V_{i,0}}) d\epsilon_{V_{i,0}}. \tag{99}$$

The Probability of Satisfying both Condition 1 and Condition 2

The probability of satisfying both Condition 1 and Condition 2 is the product of both probabilities. This probability is simply the probability that investor i purchases bond B_0 given his characteristics:

$$Pr_i(B_0|\xi_i, \eta_i, \beta_i) = Pr_i(B_0|b_{l,0}, \xi_i, \eta_i, \beta_i, \forall l \neq 0) Pr_i(B_0|B_{free}, \xi_i, \eta_i, \beta_i). \tag{100}$$

3.5.2 The Discrete/Continuous Choice of the Investor

We follow the procedure used by Hannemann (1984), Chintagunta (1993), and Richards (2000) to investigate the discrete-continuous choice of the investor. The derivation in the green bond case is similar to that in the conventional case under risk aversion. The only difference is that the optimal quantity of bonds purchased also depends on the prosocial attitude and the environmental target announced by the issuer ($\rho * \hat{S}^*_{l,0}$).

Solving problem (78) generates the optimal quantity ($n^{d,*}_{i,l_0}$) chosen by investor i for bond B_0:

$$n^{d,*}_{i,l_0} = n^{d,*}_{i,l_0}(b_0|c, \Upsilon, T, E_i(P_t), E_i(R_t), E_i(\tau_t), \alpha, \omega, T_0, \xi_i, \delta_i, \ldots \eta_i, \beta_i, b_{l,0}, b_{free}, \Gamma_C, \Gamma_V, \rho, \hat{S}^*_{l,0}) \forall l \neq 0 \tag{101}$$

3.5.3 The Aggregate Demand for the Conventional Bond Issued by a Given Firm

Suppose the market is made of risk-averse investors of type ξ_i. n^*_l is the aggregate demand of bonds averaging over the type of investors. Let g_1 be the population distribution of ξ_i, g_2 be the population distribution of β_i, and g_3 the population distribution of η_i:

$$n^{d,*}_{l,l_0} = \int\int\int Pr(B_0|\xi_i, \beta_i) n^{D,*}_{i,l_0} * g_1(\xi_i|\Gamma_{\xi_i}) g_2(\beta_i|\Gamma_{\beta_i}) g_3(\eta_i|\Gamma_{\eta_i}) \, d(\xi_i)\, d(\beta_i)\, d(\eta_i). \tag{102}$$

$n^{d,*}_{l,l_0}$ represents the market demand for bond B_0 emitted by firm l. Γ_{ξ_i} represents the parameters of the distribution of ξ_i. Γ_{β_i} represents the parameters of the distribution of β_i. In equation (102), we integrate out over the population distributions of ξ_i and β_i to compute the demand for conventional bonds (B_0) over all the alternatives by randomly selected investors.

$$n^{d,*}_{l,l_0} = n^{d,*}_{l,l_0}\left(b_0|c, \Upsilon, T, E_i(P_t), E_i(R_t), E_i(\tau_t), \alpha, \omega, T_0, b_{l,0}, b_{free}, \delta_i, \Gamma_{i,0}, \Gamma_C, \Gamma_V, \Gamma_F, \Gamma_{\xi_i}, \Gamma_{\beta_i}, \Gamma_{\eta_i}, \rho, \hat{S}^*_{l,0}\right) \text{ with } l = 0. \tag{103}$$

3.5.4 The Determination of the Equilibrium Market Price

The conventional bond section has already discussed the existence and uniqueness of price equilibrium. At equilibrium, we assume the price b_0 adjusts and is known. The optimal supply of green bonds from firm l_0 is equal to the optimal demand for green bonds for firm l_0:

$$n^{S,*}_{l_0}(b_0, c, \Upsilon, E_l(P_t), E_l(R_t), E_l(\tau_t), \alpha, \delta_l, T, T_0, \Gamma_Z, \Gamma_{IC}, \rho, \hat{S}^*_{l,0}) = n^{d,*}_{l,l_0}\left(b_0|c, \Upsilon, T, E_i(P_t), E_i(R_t), E_i(\tau_t), \alpha, \omega, T_0, b_{l,0}, b_{free}, \delta_i, \ldots \Gamma_{i,0}, \Gamma_C, \Gamma_V, \Gamma_{\xi_i}, \Gamma_{\beta_i}, \Gamma_{\eta_i}, \rho, \hat{S}^*_{l,0}\right) \text{ with } l \in \{1, \ldots, I-1\}. \tag{104}$$

Hypothesis IV (Under Risk Neutrality): On the supply side, compared to a conventional bond, the price of a green bond is affected by the cost of issuance. The hypothetical sign of the effect of those variables on the supply of green bonds can be found in Table 1.

Hypothesis V (Under Risk Aversion): On the demand side, compared to a conventional bond, the price of a green bond is additionally affected by the prosocial attitude of investors, the functional form of the utility function of investors taking into account the environmental benefit of the bond, and the environmental target set the firm. The hypothetical sign of the effect of those variables on demand for green bonds can be found in Table 1.

4 Estimation of the Greenium under Risk Aversion

Let b_0^{GB} be the equilibrium price if firm l_0 issues a green bond, and let b_0^{CB} be the equilibrium price if firm l_0 issues a conventional bond. The greenium (Gr_{i,l_0}) is the difference between b_0^{GB} and b_0^{CB} given the fact that the two bonds have similar characteristics (c, Υ, T), are used to purchased the same technology, and are emitted by the same firm for the same population (Γ_V, Γ_ξ, $\Gamma_{\theta i}$, $\Gamma_{\eta i}$), and same financial market environment (ω, risk-free assets).

$$Gr_{i,l_0} = b_0^{GB} - b_0^{CB},$$

$$Gr_{i,l_0} = Gr_{i,l}(b_{l,0}^{CB}, b_{l,0}^{GB}, \delta_i, \Gamma_{i,0}, \Gamma_C, \Gamma_\eta, \rho, \hat{S}_{l,0}^{*}, E_{l_0}(P_t), E_{l_0}(R_t), E_{l_0}(\tau_t), \ldots \quad (105)$$
$$E_i(P_t), E_i(R_t), E_i(\tau_t), \delta_l, \delta_i, c, \Upsilon, T, \alpha, T_0, \omega, b_{free}, \Gamma_V, \Gamma_\xi, \Gamma_\theta, \Gamma_{I_{GB}}, \Gamma_F).$$

where, Gr_{i,l_0} is a function of the variables determining the greenium. Equation (105) shows that the greenium depends on factors specifically affecting the supply and the demand of conventional bonds and factors specifically affecting the supply and the demand of green bonds.

Table 1 shows hypotheses on the effect of exogenous variables on the greenium. The table shows that there are conditions under which the sign of the greenium is undetermined (+/-/0), strictly negative (-), or strictly positive (+).

Hypothesis VI Some exogenous variables produce an indefinite effect on the greenium (+/-/0). This situation happens when the change in the exogenous variable does not produce a definite effect on either the equilibrium price of the conventional bond, the equilibrium price of the green bond, or both. We note that the effect of some exogenous variables on green or conventional bond prices is undetermined when those variables move supply and demand forces in the same direction for either of the bonds. See Table 1 for exogenous variables having an indefinite effect on the greenium.

Hypothesis VII Some exogenous variables produce a strictly negative effect on the greenium (-) under two conditions: (i) When the exogenous variable increases (decreases), the equilibrium price of the conventional bond (green bond) leaving the price of the green bond (conventional bond) unchanged. (ii) When the change in the exogenous variable simultaneously increases (decreases) the equilibrium price of the conventional bond (the green bond). See Table 1 for variables having a negative effect on the greenium.

Hypothesis VIII Some exogenous variables produce a strictly positive effect on the greenium (+) under two conditions: (i) When the exogenous variable decreases (increases), the equilibrium price of the green bond (conventional bond) leaving the price of the conventional bond (green bond) unchanged. (ii) When the change in the exogenous variable simultaneously decreases (increases) the equilibrium price of the conventional bond (the green bond). See Table 1 for variables having a positive effect on the greenium.

5 Numerical Application

5.1 General Assumptions

In the face of climate change, firms in diverse sectors (agriculture, entertainment, transportation, etc.) have to make a technological investment ($k_{l,0}$) to protect themselves from weather risks. Since climate change is an unforeseen event in firms' financial planning, they must resort to private loans, bonds, and stocks to finance their protection investment. In this application, we consider firm l_0 that has decided to emit a bond (c,Y, T) to purchase protection equipment that will reduce the impact of weather fluctuations on production and therefore increase profit in the incoming years. Several assumptions need to be made on firm l_0: (i) Firm l_0, a long-sighted and risk-neutral farming company with operations in Ohio, Pennsylvania, Illinois, and Indiana (states with low irrigation), has decided to invest in an irrigation system to protect its future corn harvests from drought. To secure its operations, firm l_0 purchases an hybrid[14] The irrigation system alternates between solar power and fossil fuel because solar power is intermittent. Irrigation technology uses the solar part of the system as a priority. While uncertain weather implications for production are reduced through irrigation, a new source of risk is introduced in the name of solar radiation ϕ_R.

(ii) Firm l_0 has decided to emit a bond to finance the investment of irrigation technology. The bond has characteristics (c_0,Y_0, T). According to the manufacturer, the irrigation system has a lifetime of $T_0 = 7$ years. We assume that firm l_0 emits a bond with maturity T=5 years and coupon rate ($c_0 = c = 5\%$). Bonds are typically issued in 1000 \$ denomination, so we assume that the $Y_0 = Y = 1000$ (Investopedia, 2020).

(iii) Firm l_0 is aware that they can emit two types of bonds to finance the irrigation system: (i) a conventional bond, (ii) and a green bond.

(iv) When the firm enters the bond market, it faces firm l_1 and the government treasury (l_{gov}) issuing "risk-free" treasury bonds. Firm l_1 has currently issued a bond similar in characteristics to the one that firm l_0 intends to issue with coupon ($c_1 = c = 5\%$), face value (Y = 1000), and maturity (T=5). At the time of the bond issuance of firm l_0, the bonds of firm l_1 are sold in the market at par, meaning that b_1 = Y=1000. The conventional and green bonds are compared to the treasury bonds with $Y_{gov} = Y = 1000$ \$ and a coupon rate $c_{gov} = 2\%$. Corporate bonds are generally issued at a higher coupon rate than Treasury bonds. The Treasury bond is also sold at par $b_{gov} = 1000$ \$.

(v) The firm is located in an area with a diverse population made of investors with different levels of risk aversion (β_i), prosocial attitude (η_i), and income (ξ_i). The population distribution for each of the characteristics is normally distributed.

(vi) We assume that the firm has been in activity for several years, and credit

[14]Because the irrigation technology is hybrid and uses the solar part of the system in priority, the irrigation investment is qualified under the ICMA principles as green (ICMA, 2022).

rating agencies believe that the company has an extremely low risk of committing pecuniary default on any bond emitted, whether the bond is green or conventional. Therefore, the idiosyncratic risk of default of the firm is assumed to be null. Moreover, the investor does not have access to several information on the internal costs of the firms or even the firm's risk attitude to compute the idiosyncratic risk of pecuniary default of the firm. Therefore, the firm's default probability will be solely based on the systematic risk in the market.

(vii) When the firm emits a green bond, the firm promises to reduce its past five years' level of carbon emissions ($\hat{S}^*_{l_0,0}$) by a certain percentage ρ within the next T years. The firm is aware that it will face a reputational effect (Z) in case they fail to fulfill its promise at maturity. This reputational effect affects the profit negatively from maturity and onwards.

(viii) The goal of the firm is to know which type of bond they should emit based on the equilibrium price at which they would be able to sell the green bond (b_0^{GB}) vs. the conventional bond (b_0^{CB}), as precipitation risk, is expected to increase over time, as environmental policy is strengthened over time, and as the prosocial attitude of the population evolves. We can then infer how the greenium will evolve as demand and supply factors change over time.

(ix) The central authority recommends to use δ=7 % as a discount rate. Therefore, issuers and investors use δ as a basis to calculate their discounted profits respectively and discounted expected returns.

(x) The central authority has announced how the expected value for the stochastic variables will evolve in future periods. We assume that the issuer and the investors base their expectations on the central authority announcements.

5.2 Conventional Bond Supply Specification

5.2.1 Expected Value of Stochastic Variables

We assume that firm l_0 faces four (4) sources of risks: firm l_0 casts some doubt on the fluctuation of the precipitation rate during the growing period ($W_{g,t}$), the average temperature during the growing time ($TP_{g,t}$), the output price (P_t), and input prices (R_t). We suppose that the farmer has no doubt and trusts the forecasts of the National Weather Service (NWS) regarding the average temperature and precipitation rate during the planting and harvest periods.

Let $\phi_{R,g,t}$ be the solar radiation in the area under study. $\phi_{R,g,t}$ depends on $TP_{g,t}$, so the intermittency of solar radiation results from the fact that $TP_{g,t}$ is stochastic. If $\phi_{R,g,t} < \tilde{\phi}_{R,g}$, the irrigation technology switches from solar power to fossil fuel, where $\tilde{\phi}_{R,g}$ is defined as the threshold for the solar system to work effectively.

firm l_0 evaluates the last five years' average precipitation rate during the growing period ($\mu_{W\,g,0}$). Firm l_0 believes that $\mu_{W\,g,0}$ has been at an adequate level over the past five years. Therefore, he treats $\mu_{W\,g,0}$ as reference points for irrigation for the next T_0 periods.

The Expected Water Rate during Growing Time without Irrigation

The firm has doubts about the precipitation rate during the growing period. The precipitation rate during the growing season ($W_{g,t}$) is stochastic. Let $W^{ir=0}_{tot,g,l_0,t}$ be the total water rate available to the firm l_0 without irrigation. Under no irrigation $W^{ir=0}_{tot,g,l_0,t}$ is equal to $W_{g,t}$.

$$W^{ir=0}_{tot,g,l_0,t} \equiv W_{g,t}.$$

Since the firm is risk neutral, its provisional production function is based on $E(W_{g,t})$. We note that $\mu_{W,g,t}$ increases over time due to the increasing stock of carbon emissions (D_t) over each period. $\mu_{W\,g,0}$ is the expected average precipitation at time t =0.

$$E(W_{g,t}) = \mu_{W_{g,t}} = \mu_{W_{g,0}} - \kappa_W * D_t, \quad (106)$$
$$\mu_{W_{g,0}} = SMA_5(\{W_{g,t}\}^{t=0}_{t=-5}).$$

Where κ_W represents the rate of decrease of $\mu_{W\,g,t}$ as the stock of carbon emission (D_t) increases over time.

The Expected Water Rate during Growing Time with Irrigation

$\mu_{W\,g,0}$ can also be considered as the pre-irrigation average rainfall rate and the reference level for irrigation. The farmer is aware that $\mu_{W,g,t}$ will reduce in the next T years due to the increased carbon emissions D_t over time. Under irrigation, the total water rate available to the firm equals the precipitation plus the irrigation rate.

$$W^{ir=1}_{tot,g,l_0,t} \equiv W_{g,t} + W_{ir,t} * \vartheta_{l_0,t}.$$

$W_{ir,t}$ is multiplied by $\vartheta_{l_0,t}$ because when the flow of the irrigation investment is null (i.e., when the irrigation system becomes obsolete), the farmer cannot irrigate anymore. Moreover, the higher is $\vartheta_{l_0,t}$; the more efficient the irrigation system, so less water is needed to irrigate a given area.

Let $\mu_{W^{ir=1}_{tot,t}}$ be the post-irrigation mean total water rate at time t. As discussed in the assumption section, irrigation has for goal, under risk, to maintain the mean water rate on the field within a range of 10 % from $\mu_{W\,g,0}$:

$$E(W^{ir=1}_{tot,g,l_0,t}) = \mu_{W^{ir=1}_{tot,t}} = E(W_{g,t}) + E(W_{ir,t})\vartheta_{l_0,t},$$
$$\mu_{W\,tot,t} = \mu_{W_{g,0}} - \kappa_W * D_t + E(W_{ir,t})\vartheta_{l_0,t}, \quad (107)$$
$$st: 0.9 * \mu_{W_{g,0}} < \mu_{W^{ir=1}_{tot,t}} < 1.1 * \mu_{W_{g,0}}.$$

From condition (107), we can deduce that the average rate of irrigation of the farmer ($E(W_{ir,t})$) is bounded under risk as follows:

$$\frac{0.9 * \mu_{W_{g,0}} - \mu_{W_{g,t}}}{\vartheta_{l_0,t}} \leq E(W_{ir,t}) \leq \frac{1.1 * \mu_{W_{g,0}} - \mu_{W_{g,t}}}{\vartheta_{l_0,t}}. \quad (108)$$

The Distribution of Temperature during Growing Time

The farmer has some doubt on the temperature ($TP_{g,t}$) during the growing period. We note that $E(TP_{g,t})$ increases over time due to the increasing stock of carbon emissions (D_t) over each period. $\mu_{TP_{g,0}}$ is the expected average temperature at time t =0.

$$E(TP_{g,t}) = \mu_{TP_{g,t}} = \mu_{TP_{g,0}} + \kappa_{TP} * D_t. \quad (109)$$

Where κ_{TP} represents the rate of increase of $\mu_{TP_{g,t}}$ as the stock of carbon emission (D_t) increases over time.

The Distribution of Solar Radiation during Growing Time

The Stefan-Boltzmann law states that the rate of outward radiative energy (per m^2) emitted by an object with temperature ($TP_{g,t}$) during growing time is proportional to the 4th power of $TP_{g,t}$. The higher the temperature of an object, the greater its radiative energy output will be. The minimum solar insolation needed to generate electricity is $\tilde{\phi}_{R,g}$, which is enough to activate the solar pump. In this model, the lower is $\tilde{\phi}_{R,g}$, the more efficient is the solar power technology because it needs less energy to operate. In the initial model, we assume that $\tilde{\phi}_{R,g}$ = 500 W/m^2 is the minimum solar irradiance needed for a solar-powered pump to operate (IEEE).

$$E(\phi_{R,g,t}) = \mu_{\phi_{R,g,t}} = \sigma_{SB}(E(TP_{g,t}) + 273.15)^4, \quad (110)$$
$$\mu_{\phi_{R,g,t}} = \sigma_{SB}(\mu_{TP_{g,t}} + 273.15)^4.$$

The Stefan-Boltzmann constant (σ_{SB}) = $5.67 * 10^{-8} W m^{-2} K^{-4}$. And $TP_{g,t}$ in (°C).

The Corn Harvest Price over each Period

We collected the Chicago Board of Trade price of corn over the last ten years. We then estimate the price of corn over each period by running a 10-year simple moving average of past prices. The price of corn is highly influenced by stochastic factors affecting the beginning stocks, the imports, and the production of corn. The price of corn is also affected by the demand side, including food, seed, industrial, feed and residual, exports, and carryover stocks (ERS-USDA,2021).

$$P = E_I(P_t) = SMA_{10}(\{P_t\}^t{}_{t-1}). \quad (111)$$

Since the period used to compute the SMA is large, the expected output price over the next five years will converge towards P.

Nitrogen and Water Input Prices over each Period

We assume that input prices are fixed over the bond's maturity period but could be affected by the environmental policy ($\zeta > 0$) over the next five years. When ζ is low, the environmental policy is weak, whereas when ζ is high, the environmental policy is stringent. According to Barett (1994), higher environmental standards increase the marginal cost of inputs. Therefore, the price of nitrogen (R_N) and the price of water pumped using fossil fuels (R_W) increases with ζ.

$$R_W(\zeta) = E_I(R_{W,t})(\zeta) = SMA_{10}(\{R_{W,t}\}^t{}_{t-10})(\zeta),$$
$$R_N(\zeta) = E_I(R_{N,t})(\zeta) = SMA_{10}(\{R_{N,t}\}^t{}_{t-10})(\zeta), \quad (112)$$
$$with\ \zeta = \{1, 2, 3, ..., 10\}.$$

Since the period used to compute the SMA is large, the expected input prices over the next five years will converge towards R_W and R_N. The prices of the inputs are increased if environmental policy is increased over the next five years.

5.2.2 The Multiple Production Function without Irrigation

In this essay, we assume that the farming company produces without irrigation one good output (The corn yield, $Y^{ir=0}_{l_o,t}$) and one public output (Carbon Emissions, $S^{ir=0}_{l_o,t}$). In the theory section, we have shown that the directional distance function allows us to represent in a single equation the joint production of multi-outputs using multi-inputs when some of the outputs are public. The quadratic function is used for econometric estimation because it satisfies the constraints required for the directional distance function characteristic (Fare and Grosskopf, 2006).

Estimation of the Quadratic Directional Output Distance Function

During growth time, the inputs available to the farmer to grow corn without irrigation are (i) $N^{ir=0}_{l_o,t}$(*Nitrogen*), (*ii*)$TP_{g,t}$ (Growing Time Temperature), (iii) $W_{g,t}$ (Precipitation during growing time). The outputs are corn yield ($Y^{ir=0}_{l_o,t}$) and carbon emission ($S^{ir=0}_{l_o,t}$) as a by product of $Y^{ir=0}_{l_o,t}$. The quadratic directional output distance function with three inputs and two outputs can be written as follows:

$$Y^{ir=0}_{l_o,t} = \beta_0 + \beta_1 N^{ir=0}_{l_o,t} + \beta_2 TP_{g,t} + \beta_3 W_{g,t} + \beta_4 S^{ir=0}_{l_o,t} + \frac{1}{2}(\beta_{1,1}(N^{ir=0}_{l_o,t})^2 + \beta_{2,2} TP^2_{g,t} + \ldots$$
$$\beta_{3,3} W^2_{g,t} + \beta_{1,2} N^{ir=0}_{l_o,t} TP_{g,t} + \beta_{1,3} N^{ir=0}_{l_o,t} W_{g,t} + \beta_{2,3} TP_{g,t} W_{g,t}) + \frac{1}{2}\beta_{4,4}(S^{ir=0}_{l_o,t})^2 + \ldots$$
$$\beta_{1,4} N^{ir=0}_{l_o,t} S^{ir=0}_{l_o,t} + \beta_{2,4} TP_{g,t} S^{ir=0}_{l_o,t} + \beta_{3,4} W_{g,t} S^{ir=0}_{l_o,t} + \epsilon. \quad (113)$$

Eq. (113) is a direct specification[15] to estimate a yield curve as a function of inputs and climate factors such as temperature and precipitation. Past papers that have estimated corn yield response to nitrogen have used a quadratic yield function (Llewelyn and Featherstone,1996; Bert et al., 2007; Thorp et al., 2008; Paz et al., 1999, Batchelor et al., 2002, Link et al., 2006, Dogan et al., 2006, Miao et al., 2006). Researchers have found quadratic forms to be more suitable than linear response functions for modeling corn yield response to N (Bullock and Bullock, 1994; Cerrato and Blackmer, 1990; Bullock and Bullock, 1994; Roberts et al., 2002; Boyer et al. 2013; Laila Puntel et al., 2016). Boyer and al. (2013) and Lailai Puntel (2016) used only nitrogen rates ($N_{l_o,t}$) applied to corn and $(N_{l_o,t})^2$ in their estimation of corn yield response to nitrogen. Llewelyn et al. (1996) estimated corn yield using nitrogen rates, water rates, and the square and interaction terms of nitrogen and water rates. Long-term field experiments on corn have been undertaken in Missouri (Sandborn Field), Nebraska (Knorr-Holden), and Illinois (Morrow's plot) (Scofield Holden., 1927; Aref Wander., 1997; Bijesh et al., 2021). As demonstrated in Essay 1, yield is affected by climatic conditions at planting and harvest. Therefore, we included TP_p and W_p to eq. (113) because soil conditions at planting are affected by temperature and precipitation. Similarly, we added TP_h and W_h to capture the effect of soil conditions

[15]For indirect estimation of the crop production function, this can be achieved through the specification of appropriate dual formulations, such as the cost or profit functions (Blackborby, Primont, and Russell; Diewert 1971, 1974; Jorgenson and Lau, 1974). The indirect production function is dependent on the input prices (r), the profit functions (Π), the fixed capital (ϑ), and time t, i.e., $y(r, \pi, K, t)$. The production function can then be econometrically estimated using a translog, CES, or Lewbel (Hilmer et Holt, 2005). Since we do not have farm-level profit data, the indirect estimation will not be used for our simulation.

at harvest on yield. We focus on county-level data as representative of actual rather than experimental practice. We estimate county-level yield response to nitrogen and weather as specified in the following equation for an area with low to no irrigation (Illinois, Indiana, Ohio, and Pennsylvania):

$$Y^{ir=0}_{l_0,t} = \hat{\beta}_0 + \hat{\beta}_1 N^{ir=0}_{l_0,t} + \hat{\beta}_2 TP_{g,t} + \hat{\beta}_3 W_{g,t} + \hat{\beta}_4 S^{ir=0}_{l_0,t} + \frac{1}{2}(\hat{\beta}_{1,1}(N^{ir=0}_{l_0,t})^2 + \hat{\beta}_{2,2} TP^2_{g,t} + \hat{\beta}_{3,3} W^2_{g,t} + \hat{\beta}_{4,4}(S^{ir=0}_{l_0,t})^2 + \hat{\beta}_{1,2}[N^{ir=0}_{l_0,t} * TP_{g,t}] + \hat{\beta}_{1,3}[N^{ir=0}_{l_0,t} * W_{g,t}] + \hat{\beta}_{2,3}[W_{g,t} * TP_{g,t}]) + \hat{\beta}_{1,4} N^{ir=0}_{l_0,t} S^{ir=0}_{l_0,t} + \hat{\beta}_{2,4} TP_{g,t} S^{ir=0}_{l_0,t} + \hat{\beta}_{3,4g,t} S^{ir=0}_{l_0,t} + \hat{\beta}_5 TP_{p,t} + \hat{\beta}_{5,5} TP^2_{p,t} + \hat{\beta}_6 TP_{h,t} + \hat{\beta}_{6,6} TP^2_{h,t} + \hat{\beta}_7 W_{p,t} + \hat{\beta}_{7,7} W^2_{p,t} + \hat{\beta}_8 W_{h,t} + \hat{\beta}_{8,8} W^2_{h,t} + \epsilon. \tag{114}$$

$Y^{ir=0}_{l_0,t}$ is the county-level corn yield from 1987-2012. $Y^{ir=0}_{l_0,t}$ is available in the quick stat database of the USDA/NASS. $N^{ir=0}_{l_0,t}$ is the nitrogen rate used by each county for producing corn from 1987-2012. The county-level nitrogen rate was estimated using the procedure described by Yushu et al. (2021). They use a top-down area-based approach that allocates Nitrogen fertilizer inputs into corn-producing areas by combining state-level crop-specific nitrogen fertilizer application rates (NASS) and percentage of the area receiving N fertilizer (NASS/USDA) with the county-level proportion of crop-specific planted area (USGS). Since we do not have data on irrigation rates for corn at the county level, we focus our study on major corn producers located in counties with very low to no irrigation. Those counties are within the states of Illinois, Ohio, and Pennsylvania. In counties with no irrigation, the precipitation rate is the water rate applied to corn. Precipitation and temperature data are available from the Prism database of Oregon University.

$S^{ir=0}_{l_0,t}$ is the level of carbon dioxide equivalent per acre produced by corn farming activities. Adom et al. (2012) show that the average carbon footprint (μ_S) in the Pennsylvania, Illinois, Indiana, and Ohio region is equal to 370 gCO2e/kg [16]. Therefore, to estimate the carbon emissions within each county, we scale μ_S in proportion to the amount of grain harvested within each county.

$W_{p,t}$, $W_{g,t}$ and $W_{h,t}$ are the average precipitation rate in the area during the planting season, the growing season, and the harvest season respectively. $TP_{p,t}$, $TP_{g,t}$, and $TP_{h,t}$ are the average temperature during the planting, the growing, and the harvest season, respectively. We included $TP_{p,t}$ and $W_{p,t}$ because soil conditions at planting are affected by temperature and precipitation. Similarly, we added $TP_{h,t}$ and $W_{h,t}$ to capture the effect of soil conditions at harvest on yield. Precipitation and temperature data are available from the Prism database of Oregon University. The summary statistics of the empirical variables are available in Table 2.

Results

Equation (114) was estimated using a fixed effect model at the year and state level. The results show that the nitrogen and weather variables significantly impact county-level yield (Table 3). Results suggest with 99 % confidence that nitrogen, temperature, and rainfall negatively affect yield. That means the relationship is positive for low values of nitrogen and rainfall, but for high values, the relationship becomes negative. Moreover, there exists a negative quadratic relationship between the good

[16] Carbon Dioxide Equivalent

output (corn production, $Y^{ir=0}_{l_0,t}$) and the public output (carbon emissions, $S^{ir=0}_{l_0,t}$). This result confirms the theory of Fare et al. (2005), which affirms that the relationship between $Y^{ir=0}_{l_0,t}$ and $S^{ir=0}_{l_0,t}$ has an inverted U- shape (Figure 2 in Appendix). The model accounts for the properties of the quadratic form that imposes non-zero elasticity of substitution among factors, as well as diminishing marginal productivity as inputs and public outputs increase.

5.2.3 The Multiple Output Production Function With Irrigation

The Multiple Output Production Function With Irrigation using Solar Power

Let τ_t be the set containing the weather variables, ie $\tau_t = \{W_{p,t}, TP_{p,t}, W_{g,t}, TP_{g,t}, W_{h,t}, TP_{h,t}\}$. Let $F^{ir=0}$ be the production function without irrigation. The multiple output production function described in eq. 114 can be written in a more compact form as:

$$F^{ir=0} \equiv F^{ir=0}(N^{ir=0}_{l_0,t}, W^{ir=0}_{tot,g,l_0,t} \equiv W_{g,t}, Y^{ir=0}_{t}, S^{ir=0}_{l_0,t} | \vartheta_{i,0}, E(\tau)) = 0. \tag{115}$$

The investment in the irrigation system increases the fixed factors $\vartheta_{l_0,t}(k_{l_0,0})$ available to the farmer. The fixed cost of the irrigation system vary between $k^{min}_{l_0,0}$ to $k^{max}_{l_0,0}$. Let $N^{ir=1}_{l_0,t}$, $W^{ir=1}_{l_0,t}$, $Y^{ir=1}_{l_0,t}$, $S^{ir=1}_{l_0,t}$ be respectively the nitrogen, the total water rate during growth time, the corn output, and the carbon emissions under irrigation. We assume that irrigation increases the pre-irrigation water rates, irrigation with solar power does not change the pre-irrigation carbon emissions, irrigation increases the pre-irrigation corn output, and irrigation does not change the pre-irrigation nitrogen use.

$$\begin{aligned}
N^{ir=1}_{1,t} &\equiv N^{ir=0}_{l_0,t} \\
Y^{ir=1}_{1,t} &\equiv Y^{ir=0}_{l_0,t} + 100 * \vartheta_{l_0,t}(k_{i,0}) \\
S^{ir=1}_{1,t} &\equiv S^{ir=0}_{l_0,t} \\
W^{ir=1}_{tot,g,l_0,t} &\equiv W_{g,t} + W_{ir,t} * \vartheta_{l_0,t}(k_{i,0})
\end{aligned}$$

The multiple output production function without irrigation (eq. 114) is modified as follows to account for irrigation:

$$F^{ir=1} \equiv F^{ir=0}(N^{ir=1}_{l_0,t}, W^{ir=1}_{tot,g,l_0,t}, Y^{ir=1}_{l_0,t}, S^{ir=1}_{1,t} | \vartheta_{l_0,t}, E(\tau)) = 0 \tag{116}$$

$F^{ir=1}$ (eq. 116) is a positive vertical translation of $F^{ir=0}$ (eq. 115) along the Y output axis (Figure 1). We define $\vartheta_{l_0,t}$ with the following function:

$$\begin{aligned}
&\vartheta_{l_0,t} = ln(K_{l_0,t} + 1), \\
&\frac{\partial \vartheta_{l_0,t}}{\partial K_{l_0,t}} = \frac{1}{K_{l_0,t} + 1} > 0, \\
&\frac{\partial^2 \vartheta_{l_0,t}}{\partial K^2_{l_0,t}} = -\frac{1}{(K_{l_0,t} + 1)^2} < 0, \\
&\text{where } K_{l_0,t} > 0.
\end{aligned} \tag{117}$$

Equation (117) meets the requirements of the service flow function defined in the theory with eq. (3), and $K_{l_0,t}$ is defined with eq. (4) in the theory.

The Multiple Output Production Function With Irrigation using the Fossil fuel

Irrigation using fossil fuels is different from irrigation using solar power for two reasons: (i) The usage of irrigation increases the pre-irrigation amount of carbon emissions $S^{ir=0}_{l_0,t}$, (ii) The firm has to pay a water price (R_W). Therefore, the multiple output production function is similar to the multiple output production with solar power, except that the post-irrigation carbon emissions are higher than the pre-irrigation carbon emissions.

$$S^{ir=1}_{l=0,t} = S^{ir=0}_{l=0,t} + \frac{100}{\vartheta_{l_0,t}(k_{l_0,0})}. \tag{118}$$

In the fossil fuel case, $F^{ir=1}$ (eq. 116) becomes a positive horizontal translation of $F^{ir=0}$ along the X axis, and a positive vertical translation of $F^{ir=0}$ along the Y axis. . However, the higher the cost of the irrigation system, the more efficient the irrigation system is, so the lower will be the increase in pollution.

5.2.4 Profit Definitions

The short-run profit definitions

For t = 0 and with no irrigation

$$\Pi^{ir=0,sr}_{i,0} \equiv P * Y^{ir=0}_{l_0,0} - R_N * N^{ir=0}_{l_0,0}. \tag{119}$$

For t > 0 and using solar power ($\mu_{\phi_{R,t}} > \tilde{\phi}_R$):

$$\Pi^{ir=1,sr}_{1,l_0,t} \equiv P * Y^{ir=1}_{l_0,t} - R_N * N^{ir=1}_{l_0,t}. \tag{120}$$

For t > 0 and using fossil fuel power ($\mu_{\phi_{R,t}} \le \tilde{\phi}_R$):

$$\Pi^{ir=1,sr}_{2,l_0,t} \equiv P * Y^{ir=1}_{l_0,t} - R_N * N^{ir=1}_{l_0,t} - R_W * W_{ir,t}. \tag{121}$$

Therefore, the short-run profit with irrigation for $t > 0$ ($\Pi^{ir=1,sr}_{l_0,t}$), taking into account the switching between solar power and fossil fuel is as follows:

$$\Pi^{ir=1,sr}_{l_0,t} \equiv \Pi^{ir=1,sr}_{1,l_0,t} + (\mu_{\phi_{R,t}} < \tilde{\phi}_{Rad})\Pi^{ir=1,sr}_{2,l_0,t}. \tag{122}$$

Equation 123 says that :

$$\Pi^{ir=1,sr}_{l_0,t} = \begin{cases} \Pi^{ir=1,sr}_{1,l_0,t} & \mu_{\phi_{R,t}} > \tilde{\phi}_R. \\ \Pi^{ir=1,sr}_{2,l_0,t} & \mu_{\phi_{R,t}} \le \tilde{\phi}_R. \end{cases} \tag{123}$$

The restricted short-run profit functions

We obtain the restricted profit function at t=0 ($\Pi^{sr,*}_{ir=0,i,0}$) by maximizing $E(\Pi^{ir=0,sr}_{l_0,0})$ subject to $F^{ir=0}$ and $E(\Pi^{ir=0,sr}_{l_0,0}) > 0$. We obtain the restricted profit function at $t > 0$ by maximizing $E(\Pi^{ir=1,sr}_{l_0,t})$ subject to $F^{ir=1}$ and $E(\Pi^{ir=1,sr}_{l_0,t}) > 0$.

5.2.5 The Optimal Quantity of Bonds Supplied

We first compute the long-run profit functions following eq. 23 in the theory using the restricted short-run profits functions (eq. 119, 122). To compute $\eta_{l_o}^{s,*}$, we solve the optimization problem following the procedure described in the theory section.

5.3 Conventional Bond Demand Specification

5.3.1 Stochastic Distributions

The Population Distribution of Income

In 2020, the US Census estimated that the per capita mean income is (μ_ξ) $35,384 with a deviation of $605. Congressional research service shows that the distribution of US consumer income is right-skewed—meaning that the bulk of households are found on the left-hand side of the distribution with a smaller share of households spread out to the right, with considerably more distance (in terms of income) between them (CRS, 2021). Therefore, we assume that the income (ξ) in the population is log-normally distributed with mean $\hat{\mu}_\xi = log(35,384)$, and standard deviation ($\hat{\sigma}_\xi = log(605)$). Let g_1 be the population distribution of ξ (CRS, 2021).

$$g_1 \sim logN(\hat{\mu}_\xi, \hat{\sigma}_\xi). \quad (124)$$

The population distribution of risk aversion

In this simulation, we assume that the risk aversion coefficient ($\beta_i \in [0,1]$). The population is normally distributed with mean $\mu_{\beta_i} = 0.5$ and standard deviation (σ_{β_i} =0.08). Let g_2 be the population distribution of β_i.

$$g_2 \sim N(\mu_{\beta i}, \sigma_{\beta i}). \quad (125)$$

The Researcher's Observational Error in Investor's Indirect Utility

We assume that $\epsilon_{V_{i,o}}$ a generalized extreme value distribution (GEV) to have a multinomial logit model which allows having a closed form for the probability of bond purchase (eq. 74).

5.3.2 The Probability of Pecuniary Default

The probability of default is based on the systematic risks in the area where the firm operates at a given time t. As defined in the theory section, the probability of default depends on the parameters defining the distribution of P_t, $R_{W,t}$, $R_{N,t}$, $TP_{g,t}$, $W_{g,t}$.

$$pd_{i,O}(t|\zeta) = \frac{R_W(\zeta) + R_N(\zeta) + \mu_{TP_{g,t}} + W_{g,t}^{min} + P^{min}}{R_W^{max} + R_N^{max} + TP_{g,t}^{max} + \mu_{W_{g,t}} + P} \in (0, 1). \quad (126)$$

R_W^{max}, R_N^{max}, $TP_{g,t}^{max}$, $W_{g,t}^{min}$, P^{min} are fixed parameters. The systematic probability of default is higher when the prices of inputs increase, reflecting the systematic risk faced by all firms using nitrogen and water. The probability of default increases with drought risk during growth time, i.e., when temperature increases and precipitation decreases. This reflects the risk faced by all firms in the area under study. Finally, the probability of default is higher when the output price is lower, and input prices are higher. Since input prices are affected by environmental policy, the probability of pecuniary default is also affected by environmental policy.

5.3.3 The Indirect Utility Function

We use a quadratic indirect utility function in bond yield. This function meets the properties defined in sec.(3.2.6) regarding the indirect utility functions.

$$V_{i,I_0}(y_{i,0}) \equiv y_{i,0} + \beta'_i * (y_{i,0})^2 + (\xi_i - b_0). \quad (127)$$

β'_i is the negative of β_i, the risk aversion coefficient such that $\beta_i \geq 0$. We chose a linear income price difference specification because, as described in the theory section, the equilibrium is unique under this specification.

5.3.4 The Transaction Cost Function

The difference between the price a broker-dealer pays for a bond and the price at which it is sold to the investor is known as the bond's markup. The markup is a transaction cost. Markups are usually from about 1% - 5% of the bond's original value (Y). Bond dealers generally charge higher markups on smaller bond sales than larger ones are reduced as more bonds are purchased by the investor (Morningstar). Because the green bond market is smaller than the conventional bond, transaction costs are currently higher in the green bond market than in the conventional bond market. Therefore we assume in our initial simulation that transaction costs are higher by 4 % compared to the conventional bond market. From the characteristic of the transaction cost function given in eq. 63, it can be represented by an exponential function of the number of bonds purchased:

$$C(n^D_{i,I_0}) = [0.09 * exp(-0.002 * n^D_{i,I_0}) + \alpha_1] * Y * n^D_{i,I_0}, \text{ for } n^D_{i,I_0} > 0, \quad (128)$$
$$\alpha_1 = 0.01.$$

5.4 Green Bond Supply Specification

The stochastics affecting the supply of green bonds are similar to the stochastic factors affecting the supply of conventional bonds (sec. 3.5.2). The multiple output production functions with and without irrigation are similar to the conventional bond case. In the green bond case, one additional functions need to be characterized: The green bond issuance cost function.

5.4.1 The Green Bond Issuance Cost Function

The higher the number of bonds issued, the lower the cost of monitoring the green bonds. The issuance cost of a green bond represents 1 % to 6 % of the bond's face value, depending on the issuance size. We assume that the issuance costs of conventional bonds are negligible.

$$IC(n^s_{l_0}) = [0.076 * exp(-0.003 * n^s_{l_0}) + \alpha_1] * Y * n^s_{l_0} \text{ for } n^s_{l_0} > 0,$$
$$IC(k_{l_0,0}) = [0.076 * exp(-0.003 * \frac{k_{l_0,0}}{b_0}) + \alpha_1] * Y * \frac{k_{l_0,0}}{b_0} \text{ for } b_0 \neq 0, \quad (129)$$
$$\alpha_1 = 0.01.$$

5.4.2 The Optimal Quantity of Green Bonds Supplied

After expressing the green bond cost of issuance function, we compute the long-run profit functions following eq. 89 and 90 in the theory using the restricted short-run profits functions (eq. 119, 122). To compute $n^{s,*}_{l_0}$, we solve the optimization problem following the procedure described in the theory section.

5.5 Green Bond Demand Specification

5.5.1 Stochastic Distributions

The distributions of risk aversion, income, and observational error defined in the conventional bond case remain the same in the green bond case. In the case of the green bond, we also need to define the distribution of prosocial attitudes.
Prosocial attitude is defined by η_i in [0,1] such that the lower is η_i, the lower the prosocial attitude of the investor toward the environment.

The Distribution of Prosocial Preferences
For the initial simulation, we assume that the distribution (g_3) of prosocial attitude is normal with $\mu_{\eta_i} = 0.5$ and $\sigma_{\eta_i} = 0.08$.

$$g_3 \sim N(\mu_{\eta i}, \sigma_{\eta i}). \quad (130)$$

5.5.2 The Indirect Utility with Prosocial Preferences

We use an indirect utility which is a quadratic indirect utility function in bond yield and cubic in the size of the environment benefit.

$$V_{i,l_0}(y_{i,l_0}, J_{l_0,T}) \equiv y_{i,0} + \beta'_i * (y_{i,0})^2 + \eta'_i * (J_{l_0,T})^3 + (\xi_i - b_0). \quad (131)$$

A cubic specification for the environmental impact was chosen as it allows to maintain the sign of the environmental impact while showing the magnitude of the importance of J_{l_0}, T for the investor. A quadratic specification would have changed the sign of the environmental impact. where β'_i is the negative of β_i, the risk aversion coefficient as defined in 3.2.3. η'_i is the negative of η_i, the continuous variable representing the investor's prosocial attitude level. If $J_{l_0,T}$ is positive (green default), then $J_{l_0,T}$ has a negative impact on the utility of the investor. Whereas, if $J_{l_0,T}$ is negative (Green promise fulfilled), then $J_{l_0,T}$ has a positive impact on the utility of the investor. We chose a linear income price difference specification because, as described in the theory section, the equilibrium is unique under this specification.

5.5.3 The Transaction Cost Function

The difference between the price a broker-dealer pays for a bond and the price at which it is sold to the investor is known as the bond's markup. The markup is a transaction cost. Markups are usually from about 1% - 5% of the bond's original value (Y). Bond dealers generally charge higher markups on smaller bond sales than larger ones are reduced as more bonds are purchased by the investor (Morningstar). Because the green bond market is smaller than the conventional bond, transaction costs are currently higher in the green bond market than in the conventional bond market. Therefore we assume in our initial simulation that transaction costs are higher by 4 % compared to the conventional bond market. From the characteristic of the transaction cost function given in eq. 63, it can be represented by an exponential function of the number of bonds purchased:

$$C(n^{D}_{i,l_0}) = [0.1384 * exp(-0.002 * n^{D}_{i,l_0}) + \alpha_2] * Y * n^{D}_{i,l_0} \text{ for } n^{D}_{i,l_0} > 0, \quad (132)$$
$$\alpha_2 = 0.04.$$

6 Simulation Algorithm

The simulation was conducted in Matlab (Version R2020a). The goal of the simulation is to compare, under several circumstances, the equilibrium price of a conventional bond (b_0^{CB}) vs. the equilibrium price of a green bond (b_0^{GB}) with similar characteristics and intended purpose: (i) The difference between b_0^{GB} and b_0^{CB} when drought during growing time is expected to increase over time, i.e., lower expected precipitation rate and higher expected temperature during the growing time.
(ii) The difference between b_0^{GB} and b_0^{CB} for a different level of environmental regulation after the bond issuance ($\zeta = 1$ to $\zeta = 5$).
(iii) The difference between b_0^{GB} and b_0^{CB} for different levels of the mean prosocial attitude of the population (hedonic to prosocial).
(iv) The difference between b_0^{GB} and b_0^{CB} for different levels of solar technology efficiency (lowering the minimum solar radiation necessary to operate the solar system).
(v) The difference between b_0^{GB} and b_0^{CB} for reduced costs of green bond issuance.

In the following simulations, we will need to obtain the demand and supply curves for bonds. The supply curve for bonds needs to satisfy the law of supply, which asserts that, all other factors being equal, as the price of a good or service increases, the number of goods or services suppliers offer will increase, and vice versa. The demand curve needs to satisfy the law of demand, which asserts that the quantity of goods purchased varies inversely with price. Therefore, in this simulation, we define BP as the set where the supply and demand curves are defined for each simulation.

Simulation I: Change in Greenium for an Increase in Climate-related Risks
The simulation is divided into five (5) steps: **Step 1** Determination of the supply curve if the firm issues a conventional bond: (i) Initiate Parameters (Climate variables, Bond parameters, Input, and Output Prices)
(ii) Create a vector containing different b_0^{CB} prices
(iii) For b_0^{CB} in set BP

Solve eq.(30) for each b_0^{CB},
Save $n_{I_0}^{*,s}$ (eq. 33)
end

(iv) Fit a curve to obtain the relationship between $n_{I_0}^{*,s}$ and b_0^{CB}. This curve is the conventional bond supply curve

Step 2 Determination of the aggregate demand curve if the firm issues a conventional bond: (i) Initiate Parameters (Climate variables, Bond Parameters, Input Prices, distributions of investor's characteristics)
(ii) Create a vector containing different b_0^{CB} prices
(iii) for b_0^{CB} in set BP
Solve eq.(83) for each b_0^{CB},
Save $n_{I,I_0}^{d,*}$ (eq. 84)
end

(iv) Fit a curve to obtain the relationship between $n_{I,I_0}^{d,*}$ and b_0^{CB}. This curve is the conventional bond demand curve.

Step 3 Finding the equilibrium price: Solve the equation $n_{I,I_0}^{d,*}(b_0^{CB}) = n_{I,I_0}^{d,*}(b_0^{CB})$ for $b_0^{CB,*}$

Step 4 Repeat steps 1 to 3 for the green bond and estimate $b_0^{GB,*}$: Solve the equation $n_{I,I_0}^{d,*}(b_0^{GB}) = n_{I,I_0}^{d,*}(b_0^{GB})$ for $b_0^{GB,*}$

Step 5 Estimate the greenium $b_0^{GB,*}$ - $b_0^{CB,*}$ for increasing expected drought risk (lower expected precipitation)
for κ_W in ($0 : 10^{-2}10^{-1}$) (eq. 106, and 107)
Repeat step 1 to 4 in simulation 1
end

Simulation II: Change in Greenium for Different Levels of Environmental Regulation

The simulation is divided into five steps:

Step 1 to Step 4 Similar to Simulation 1

Step 5 Estimate the greenium $b_0^{GB,*}$ - $b_0^{CB,*}$ for a different level of the environmental standard after the bond issuance (ζ). ζ affects the nitrogen and the price of irrigation water pumped using fossil fuels. (eq. 112)
for ζ in (1 : 1 : 5)
Repeat step 1 to 4
end

Simulation III: Change in Greenium as Prosocial Attitude Increase in the Population

The simulation is divided into five (5) steps:

Step 1 to Step 4 Similar to Simulation 1

Step 5 Estimate the greenium $b_0^{GB,*} - b_0^{CB,*}$ for different level of investors' average prosocial attitude (η_i) (eq. 130, 131). η_i affects the utility function of the investors (eq.132)
for η_i in (0 : 1 : 10)
Repeat step 1 to 4
end

Simulation IV: Change in Greenium for Different Levels of Irrigation Technology Efficiency

The simulation is divided into five (5) steps:

Step 1 to Step 4 Similar to Simulation 1

Step 5 Estimate the greenium $b_0^{GB,*} - b_0^{CB,*}$ for different levels of solar technology efficiency ($\tilde{\phi}_R$)

for $\tilde{\phi}_R$ in (100 : 100 : 600)

Repeat step 1 to 4 in Simulation 1

end

Simulation V: Change in Greenium for Increasing Levels of the Cost of Green Bond Issuance

The simulation is divided into five (5) steps:

Step 1 to Step 4 Similar to Simulation 1

Step 5 Estimate the greenium $b_0^{GB,*} - b_0^{CB,*}$ for different cost of green bond issuance (IC)

for α_1 in (0.01 : 0.05) (eq. 129)

Repeat step 1 to 4 in Simulation 1

end

7 Numerical Simulation Hypotheses

Simulation I: Change in Greenium for an Increase in Climate-Related Risks

Expected increased levels of drought will more likely increase the supply of bonds (both green and conventional) in the market as the firm will need to purchase an efficient irrigation system to mitigate the effects of the drought. On the demand side, we can expect the drought to reduce the demand for bonds as the probability of default of the firm increases as the drought increases. Therefore, we can expect the price of both green and conventional bonds to decrease. A decrease in both the price of the green bond and the conventional bond due to drought leaves the sign of the greenium indefinite. The change direction of the greenium depends on the size of the reduction in the green bond's equilibrium price vs. the conventional bond's equilibrium price. (See Table 1).

Hypothesis I The effect of drought on the greenium is indefinite because the direction of the change depends on the size of the reduction in the equilibrium price of the green bond vs. the equilibrium price of the conventional bond.

Simulation II: Change in Greenium for Different Levels of Environmental Regulation

Increased environmental regulation will increase the price of inputs which will more likely reduce the demand for conventional bonds because of the increase in the probability of pecuniary default of the firm. As to the supply of conventional bonds, several scenarios could occur: (1) the firm may reduce its supply of bonds from fear of not being able to pay back the investors, (2) the firm may increase its supply of bonds to purchase the hybrid irrigation system that will allow the firm to avoid paying for water when the irrigation system is in solar power mode.

Hypothesis II The effect of input prices on the greenium is indefinite as the effect on the equilibrium price of the conventional bond is ambiguous (Table 1).

Simulation III: Change in Greenium as Prosocial Attitude Increase in the Population

As prosocial attitude increases in the population, the demand for green bonds will increase. The increase in the prosocial attitude does not directly affect the supply of green bonds. Therefore, we can predict that an increased prosocial attitude in the population will increase the price of green bonds. As to conventional bonds, an increase in the prosocial attitude of the population will more likely reduce or leave the demand for conventional bonds unchanged. Similar to the green bond issuer, an increase in the prosocial attitude does not directly affect the supply of conventional bonds. Therefore, we can predict that the price of green bonds will increase, whereas conventional bonds will reduce or not change.

Hypothesis III The effect of prosocial attitude on the greenium is positive (Table 1).

Simulation IV: Change in Greenium for Different Levels of Irrigation Technology Efficiency

Increasing the technical efficiency of the hybrid technology will increase the demand for green bonds because investors will expect to receive high green benefits at the maturity of the bonds Moreover, the supply of green bonds will increase as the firm expect to receive the green reputational effect at maturity. Therefore, the impact of the increased efficiency technology on the equilibrium price of green bonds depends on the increase in demand relative to the increase in supply.

Hypothesis IV The effect of increased technology efficiency on the green bond is ambiguous as the effect of efficient technology on the price of a green bond is indefinite (Table 1).

Simulation V: Change in Greenium for Increasing Levels of the Cost of Green Bond Issuance

A reduction in the cost of green bond issuance will increase the supply of green bonds, therefore reducing the price of green bonds, all else equal.

Hypothesis V The effect of the cost of issuance of green bonds on the greenium is negative as the equilibrium of the green bond will reduce.

8 Numerical Results

The results for the numerical simulations are summarized in tables 7,8,9,10, 11, and 12. These tables share the same structure: (i) they contain eight (8) key statistics, (ii) they contain 5 cases illustrating the change in the parameter varying during each simulation.

8.1 Key Statistics of the Simulations

In this section, we explain the key statistics generated for each simulation. These statistics are illustrated using graphics and results from the base scenario. The parameters used to generate the base scenario are available in Table 5.

The Average Supply of Conventional Bonds (Avg_sup_conv)

Avg_sup_conv is a statistic that is derived from the estimation of the supply curve of the firm issuing the conventional bond. To obtain this supply curve, we use the procedure described in the producer theory under conventional bond issuance to estimate the optimal amount of bonds ($n^{s,*}_{l,conv}$) emitted by the firm at each possible bond price ($b_{l,0}$) that the firm could offer. For each simulation, the potential prices the firm could offer are the prices over which the firm's supply curve is defined (sloping upward). For the base scenario (Figure 2), we considered $N_b = 380$ possible prices the firm could offer. The dots in figure 2 represent the optimal quantities for the set of potential prices offered by the firm in the base scenario.

The average supply of conventional bonds that the firm could offer is the sum of the number of bonds offered by the firm at each possible price, divided by the number of possible prices considered (N_b):

$$Avg\_sup\_conv = \frac{\sum_{k=1}^{k=N_b} n^{s,*}_{l,conv}(b_{l,k})}{N_b}. \tag{133}$$

where k is the index of a given possible price the firm offers.

The statistic Avg_sup_conv allows us to quantify the shifts of the conventional bond supply curve relative to the base scenario for each case.

The Average Supply of Green bonds (Avg_sup_green)

Avg_sup_green is a statistic that is derived from the estimation of the supply curve of the firm issuing a green bond. To obtain this supply curve, we use the procedure described in the producer theory, the optimal amount of bonds ($n^{s,*}_{l,green}$) emitted by the firm at each possible bond price ($b_{l,0}$) that the firm could offer. The statistic Avg_sup_green is derived similarly to that we derived Avg_sup_conv but using the green bond supply curve. Figure 3 provides the supply curve of green bonds for the base scenario. The dots in figure 3 represent the optimal quantities for the set of possible prices the firm offers in the base scenario. These quantities were used to compute Avg_sup_green.

$$Avg\_sup\_green = \frac{\sum_{k=1}^{k=N_b} n^{s,*}_{l,green}(b_{l,k})}{N_b}. \tag{134}$$

The statistic Avg_sup_green allows us to quantify, for the different cases run during each simulation, the shifts of the green bond supply curve relative to the base scenario.

The Average Demand of Conventional Bonds (Avg_dem_conv)

Avg_dem_conv is a statistic derived from the conventional bond demand curve. To obtain this demand curve, we use the procedure described in the multinomial/discrete choice demand theory for conventional bonds, the optimal amount of conventional bonds demanded by investors ($n^{d,*}_{l,conv}$) at each possible bond price ($b_{l,0}$) they are willing to pay. The dots in figure 4 represent the optimal quantities demanded by the population of investors for the set of potential prices they are willing to pay in the base scenario.

The average demand of conventional bonds is the sum of the number of bonds demanded by the population of investors at each possible price, divided by the number of possible prices considered (N_b):

$$Avg\_dem\_conv = \frac{\sum_{k=1}^{k=N_b} n^{d,*}_{l,conv}(b_{l,k})}{N_b}. \tag{135}$$

The statistic $Avg\ dem\ conv$ allows us to quantify, for the different cases run during each simulation, the shifts in the conventional bond demand curve relative to the base scenario.

The Average Demand of Green bonds (*Avg_dem_green*).

$Avg\ dem\ green$ is a statistic that is derived from the estimation of the demand curve of the firm issuing a green bond. To obtain this demand curve, we use the procedure described in the multinomial/discrete choice demand theory for green bonds to estimate the optimal amount of green bonds ($n^{d,*}_{l,green}$) demanded by investors at each possible bond price ($b_{l,0}$) they are willing to pay. The statistic Avg_dem_green is derived similarly to that we derived Avg_dem_conv but using the green bond demand curve. Figure 5 provides the demand curve of green bonds for the base scenario.

$$Avg\_dem\_green = \frac{\sum_{k=1}^{k=N_b} n^{d,*}_{l,green}(b_{l,k})}{N_b}. \tag{136}$$

The statistic $Avg\ dem\ green$ allows us to quantify, for the different cases run during each simulation, the shifts in the green bond demand curve relative to the base scenario.

The Conventional Bond Equilibrium Price (*$Equi\_price\_conv$*)

Equi_price_conv is derived by finding the intersection point of the supply curve and the demand curve for conventional bonds. As described in the algorithm section, the supply curve is derived by calculating the optimal quantity supplied for each possible price. Therefore, to obtain a functional form for the conventional bond supply curve, we fit a curve between $n^{s,*}_{l,conv}$ and $b_{l,0}$. This fitted curve allows to have an estimate ($\hat{n}^{s,*}_{l,conv}$) of $n^{s,*}_{l,conv}$ as a function of $b_{l,0}$. Figure 6 shows the fitted supply curve and the base scenario's actual supply curve. In the base scenario, the curve that best fits the actual supply curve has an exponential shape.

Similarly, a functional form for the demand curve is obtained by fitting a curve between $n^{d,*}_{l,conv}$ and $b_{l,0}$. This fitted curve allows to have an estimate of $n^{d,*}_{l,conv}$ ($\hat{n}^{d,*}_{l,conv}$) as a function of $b_{l,0}$. Figure 8 shows the fitted conventional bond demand curve together with the actual demand curve for the base scenario.

We then estimate the conventional bond equilibrium price by finding the intersection point of intersection of the fitted curves. Mathematically, the estimate of the equilibrium price is obtained by equaling $\hat{n}^{d,*}_{l,conv}$ and $\hat{n}^{s,*}_{l,conv}$ and solving for $Equi\widehat{\_price\_}conv$.

$$\hat{n}^{d,*}_{l,conv}(Equi\widehat{\_price\_}conv) = \hat{n}^{s,*}_{l,conv}(Equi\widehat{\_price\_}conv). \tag{137}$$

It is important to note that the conventional bond equilibrium prices in tables 7,8,9,10, 11, and 12 are the estimated equilibrium prices based on the fitted curves (eq. 136), not the actual ones. See Figure 10 for the determination of the equilibrium price of the conventional bond equilibrium price of the base scenario.

Equilibrium Price of the Green Bond (*Equi_price_green*)
We estimate the equilibrium price of the green bond in a similar fashion. We derived the estimate of the equilibrium price in the conventional case but using the estimated green bond supply and demand curves. Mathematically, The estimate of the green bond equilibrium price can be obtained by equaling $\hat{n}^{d,*}_{l,green}$ and $\hat{n}^{s,*}_{l,green}$ and solving for $Equi\widehat{\_price\_}green$. Figures 7 and 9 show the fitted green bond supply and demand curves.

$$\hat{n}^{d,*}_{l,green}(Equi\widehat{\_price\_}green) = \hat{n}^{s,*}_{l,green}(Equi\widehat{\_price\_}green). \tag{138}$$

It is important to note that the green bond equilibrium prices in tables 7,8,9,10, 11, and 12 are the estimated equilibrium prices based on the fitted curves (eq. 138), not the actual ones. See Figure 11 for the determination of the green bond equilibrium price of the base scenario.

Statistic VII: The Environmental Default of the Firm (*Env_default*)
The environmental default of the firm is computed by finding the difference between the total amount of carbon emitted during the period of maturity of the green bond vs. the environmental target set by the firm at maturity ($\hat{S}_{l,T}$).

$$Env\_default(b_l) = \frac{\sum_{t=1}^{t=T} S^*_{l,t}(k^*_{l,0}(b_l))}{T} - \hat{S}_{l,T}. \tag{139}$$

Since *Env_default* depends on the price of the bond, the statistic $Env\widehat{\_default}$ presented in tables 7,8,9,10, 11, and 12 is the environmental default at the estimated equilibrium green bond price ($Equi\widehat{\_price\_}green$).

Statistic VIII: The Greenium
The statistic $\widehat{Greenium}$ is defined as the difference between the green bond's estimated equilibrium price vs. the conventional bond's estimated equilibrium price.

$$\widehat{Greenium} = Equi\widehat{\_price\_}green - Equi\widehat{\_price\_}conv. \tag{140}$$

8.2 The Statistics for the Base Scenario

We computed the key statistics for the base scenario (Table 5). The base scenario shows the existence of a positive greenium between the green and the conventional bond issued by the firm. The magnitude of this greenium is equal to 55 USD. This can be due to climate, input and output prices, and technology efficiency. The demand for green bonds in the base scenario is higher than for conventional bonds, and this could be explained by the presence of investors with prosocial preferences. Moreover, the average supply of green bonds is higher than the average supply of conventional bonds. This could be explained by the fact that the green bond issuer needs to emit more green bonds to compensate for the issuance cost. Given the fact that the existence of the greenium in the base scenario could be explained by different causes, we undertake the following simulations to identify the relationship between the greenium and the following parameters: (i) the precipitation rate and the temperature during the growing time, (ii) The price of nitrogen, (iii) The level of prosocial preference within the investor population (iv) The efficiency of green technology, (v) The cost of green bond issuance.

8.3 Simulation I: The Change in Greenium for an Increase in Climate-related Risks

The Change in Greenium for a Decrease in Precipitation Rate during Growing Time

In the base scenario (Case 1), the precipitation rate during growing time decreases by 1.25 cm every year. Therefore, this simulation seeks to understand how the greenium would change if the precipitation rate during growing time decreases every year by 2.5 cm (Case 2), 3.75 cm (Case 3),5 cm (Case 4), and 6.25 cm (Case 4). The results show that the precipitation rate has a non-linear effect on the greenium. The greenium decreases by 14 % when the precipitation rate decreases by 2.5 cm. However, the greenium restarts to grow for larger decreases in precipitation rate. Table 7 shows that a decrease in precipitation rate by 6.25 cm increases the greenium by 5.5 % relative to the base scenario. The non-linear effect of precipitation on the greenium can be explained by the shifts in the supply and the demand of the two types of bonds. As shown in Table 7, a decrease in precipitation rate by 1.25 cm reduces the demand for conventional bonds by 3.8 % and reduces the demand for green bonds by 14 %. This is because a decline in precipitation increases the firm's default risk. The risk of default of the firm increases because the profit of the firm is expected to be reduced due to low precipitation. Despite the reduction in the demand for both bonds, we observe an increase in the greenium for lower levels of precipitation because the decline in the equilibrium price of the green bond is much smaller than the decline in the equilibrium price of the conventional bond (Table 7,8).

The same non-linear effects are observed when the temperature is increased relative to the base scenario (Table 9,10). An increase in temperature by 1.25 $^{\circ}$C leads to an increase in the greenium by 74 %. However, an increase in temperature by 2.5 cm leads to a relatively lesser increase in the greenium (12%). The demand for green and conventional bonds decreases as temperature increases at higher rates. While the average supply of green bonds is reduced, the average supply of conventional bonds increases. The average supply of conventional bonds increases because the firm needs more investment to mitigate the temperature rise. The increase in the supply of conventional bonds combined with the decrease in the demand for conventional bonds leads to a decrease in the conventional bond equilibrium price by 37 %. As shown in Table 10, the increase in temperature leads to an increase in the size of the firm environmental default by 17 %. The demand for green bonds is reduced because of the environmental default of the firm. With the rise of temperature, the green bond issuer does not supply because he anticipates that he would not be able to commit to the environmental performance criteria required to attract demand for the green bonds.

Overall, as discussed in hypothesis I, climate variables do not have a monotonic effect on the greenium. The effect could be negative or positive depending on the magnitude of the climatic change.

8.4 Simulation II: The Change in the Greenium for Different Levels of Environmental Regulation

In the base scenario, the cost of nitrogen is set at 1.3 $/kg. In this simulation, we increase the price of nitrogen relative to the base scenario by 100 % (2.6 $/kg) in case 2, 200 % (3.9$/kg) in case 3, 400 % (5.2$/kg) in case 4, 500 % (6.5 $/kg) (Case 5). We assume that input prices will likely get higher as environmental regulation strengthens. Therefore, environmental regulation is varied through the price of the inputs. Environmental regulation is considered low when input prices are low and the opposite when input prices are high. The results show that environmental regulation directly reduces the demand for conventional bonds due to the increased probability of pecuniary default of the issuer (Table 11). However, as seen in Table 11, the higher the environmental policy, the smaller the size of the environmental default committed by the firm. Therefore, green bonds attract a new group of investors with environmental preferences, so there is a higher demand for green bonds. The table shows that increasing environmental policy increases the greenium through a higher price for green bonds. As the price of nitrogen increases by 1.3 ($/kg) in the base scenario to level, the greenium increases by 563 USD (Tables 11, 12).

The positive greenium here is due to that green default decreases as the firm is constrained to use fewer inputs for production. When the green default is reduced, prosocial investors give a higher value to the green bond than conventional bonds.

8.5 Simulation III: The Change in the Greenium for Different Levels of Investor Prosocial Preferences

In this simulation, we increase the mean of the distribution of prosocial preference from 0.5 to 1. This allowed us to estimate how investors with prosocial preferences value green bonds. We conducted two simulations: (i) with a high environmental promise and (ii) with a low environmental promise. Under a high environmental promise, the firm is more likely to commit green default, as shown in the table, i.e., it has produced more than it promised. Therefore, as seen in the table, the higher the prosocial attitude of the investor, the lower the greenium when the firm commits green default. The greenium decreased by 4 USD the prosocial preferences increased from 0.5 to 1 (Tables 13, 14).

8.6 Simulation IV: The Change in the Greenium for Different Levels of Irrigation Technology Efficiency

In this simulation, we vary the efficiency of the irrigation technology by varying the minimum level of solar radiation needed for the solar system of the irrigation technology to operate. We assume this parameter is exogenous to the firm and cannot be controlled through investment. The result shows that improving the efficiency of the solar irrigation system by reducing the amount of solar radiation needed to operate from 600 Wm^{-2} to 400 Wm^{-2} allows for increasing the greenium by 10 USD (Tables 15, 16). Once the minimum solar radiation needed to operate is below 400 Wm^{-2}, we see no further changes in the greenium. The greenium increases as the technical efficiency is improved because the solar part of the irrigation system is used more than the fossil fuel. When solar technology is used, less carbon is emitted by the firm, reducing the probability of the firm's green default. This is the reason we see an increase in the equilibrium price of green bonds relative to conventional bonds as the efficiency of the irrigation technology increases (Tables 15, 16).

8.7 Simulation V: Change in Greenium for Increasing Levels of the Cost of Green Bond Issuance

In this simulation, we increase the issuance cost from 1 % to 15 % of the bond issuance. The result shows that the cost of issuance increases the greenium by reducing the supply of green bonds, leaving the demand for green bonds unchanged and the supply and demand for conventional bonds unchanged. When the issuance cost increases by 4 %, the greenium increases by 17 USD. (Tables 17, 18. The cost of issuance reduces the supply of bonds, which leads to a higher equilibrium price for the green bonds relative to the conventional bond (Tables 17, 18).

9 Policy Contributions

Several institutions such as the IMF and the World Bank are looking to understand the difference in the price of green bonds vs. conventional bonds (the greenium) and find strategies to increase the market quantity of green bonds. From the result of our simulations, several recommendations could be made to achieve these objectives:

The Need to increase the Investment in Green Technology Research

Simulation IV shows that improving the efficiency of green technology has the potential to increase the demand for green bonds simultaneously. This is because improving the efficiency of green technology reduces the possibility of green default, which is beneficial for the investor purchasing the bond.

The Need to Strengthen Environmental Regulation

Simulation II shows that a stronger environmental policy has the potential to increase the demand for green bonds. This is because environmental policy forces the issuer to reduce its carbon emissions by controlling the usage of polluting inputs. The demand increases because investors anticipate that the firm is unlikely to commit green default given the constraints imposed by the government on the price of carbon emissions and the price of inputs.

The Need to Inform the Public about Climate Change
Simulation I shows prosocial investors purchase more green bonds than hedonic investors when they estimate that the firm issuing the bonds has a low probability of green default. This is why it is important to make green technology more efficient and strengthen environmental regulation to reduce the probability of green default. If the prosocial investor does not trust the green technology proposed by the green bond issuer, they will not purchase the green bond. Moreover, suppose environmental regulation does not encourage issuers to demonstrate sustainable behavior. In that case, prosocial investors will not invest in green bonds as they anticipate that the issuer is likely to commit green default without being penalized. The public must understand that green default is not always the firm's decision. It could occur due to random processes such as input and output price fluctuations and climatic fluctuations during the bond's maturity period.

The Need for Issuers to be Truthful about Their Environmental Promise
Issuers need to give reasonable environmental targets when issuing green bonds. Simulation III shows that green bond issuance is unlikely to attract prosocial investors if they expect the firm to be unable to commit to its environmental promise. This can be achieved through regulation, given knowledge of the existing green technology, regarding the information communicated by the firm during a green bond issuance.

The Need to Reduce the Cost of Green Bond issuance
Issuers need to give reasonable environmental targets when issuing green bonds. As shown in the simulation V, higher issuance cost decreases the supply of green bonds in the market. Therefore, if we need more green issuance, there is a need to simplify the process of green bond issuance for firms.

10 Conclusion

In this essay, we developed a theory to understand the difference in the market price of green vs. conventional bonds at issuance. The theory showed us that the supply and the demand of these bonds at issuance determine the market price of bonds at issuance. We applied this theory to a farmer deciding whether to emit green vs. conventional bonds to finance a green investment. The theory allowed us to determine the optimal quantity of conventional and green bonds the firm will be willing to emit. Investors will be willing to purchase under five scenarios during the period of maturity of the bond: (i) The increase in the temperature rate and a decrease in precipitation rate during the growing time, (ii) The strengthening of the environmental policy during the maturity period of the bond, (iii) The change in the prosocial attitude of investors, and (iv) The improvement of the efficiency of the green technology used by the issuer, (v) The increase in the cost of green bond issuance.
Compared to conventional bonds, the theory shows that the demand for green bonds is influenced not only by the probability of financial default but by the probability of green default of the issuer.

The simulations show that the probability of green default is influenced by the fluctuations in weather (precipitation, solar radiation), the efficiency of the green technology used by the issuer, and the strength of the environmental policy regarding carbon emissions. The simulations show that establishing a strong environmental policy and reliable green technology is a sine qua none to developing the green bond market. Without these pre-conditions, even prosocial investors will not be willing to purchase green bonds as they anticipate that the firm will commit green default, which will decrease the value of their bonds.

We recommend the following strategies to boost the green bond market: (i) The increase in the investment in green technology research to increase the reliability of green systems. (ii) The strengthening of environmental policy to force issuers to use green systems and penalize green bond issuers failing to comply with their promise of carbon emission reduction. (iii) The sensitization of the public to the protection of the environment to increase the proportion of investors with a highly prosocial attitude in the population, iv) Finally, the regulation of the green bond issuance process to reduce the cost of issuance and force the firm to communicate realistic environmental target to the potential investors.

References

(n.d.).

Abdellaoui, M., & Wakker, P. P. (2005). The likelihood method for decision under uncertainty. *Theory and Decision*, *58*. doi: 10.1007/s11238-005-8320-4

Aimin, H. (2010). Uncertainty, risk aversion and risk management in agriculture. *Agriculture and Agricultural Science Procedia*, *1*. doi: 10.1016/j.aaspro.2010.09.018

Akter, S., Krupnik, T. J., & Khanam, F. (2017). Climate change skepticism and index versus standard crop insurance demand in coastal bangladesh. *Regional Environmental Change*, *17*. doi: 10.1007/s10113-017-1174-9

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'ecole americaine. *Econometrica*, *21*. doi: 10.2307/1907921

Aref, S., & Wander, M. M. (1997). Long-term trends of corn yield and soil organic matter in different crop sequences and soil fertility treatments on the morrow plots. *Advances in Agronomy*, *62*. doi: 10.1016/S0065-2113(08)60568-4

Artzner, P., Delbaen, F., Eber, J. M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, *9*. doi: 10.1111/1467-9965.00068

Babcock, B. A. (2015). *Using cumulative prospect theory to explain anomalous crop insurance coverage choice* (Vol. 97). doi: 10.1093/ajae/aav032

Baillon, A., Bleichrodt, H., Keskin, U., & ... (2013). *Learning under ambiguity: An experiment using initial public offerings on a stock market.*

Baillon, A., Liu, N., & van Dolder, D. (2017). Comparing uncertainty aversion towards different sources. *Theory and Decision*, *83*. doi: 10.1007/s11238-016-9584-6

Barberis, N. C. (2013). *Thirty years of prospect theory in economics: A review and assessment* (Vol. 27). doi: 10.1257/jep.27.1.173

Bard, S. K., & Barry, P. J. (2001). Assessing farmers' attitudes toward risk using the "closing-in" method. *Journal of Agricultural and Resource Economics*, *26*.

Batchelor, W. D., Basso, B., & Paz, J. O. (2002). Examples of strategies to analyze spatial and temporal yield variability using crop models. In (Vol. 18). doi: 10.1016/S1161-0301(02)00101-6

Ben-Tal, A., & Hochman, E. (1985). Approximation of expected returns and optimal decisions under uncertainty using mean and mean absolute deviation. *Zeitschrift für Operations Research*, *29*. doi: 10.1007/BF01918761

Ben-Tal, A., & Teboulle, M. (1986). Expected utility, penalty functions, and duality in stochastic nonlinear programming. *Management Science*, *32*. doi: 10.1287/mnsc.32.11.1445

Bert, F. E., Laciana, C. E., Podestá, G. P., Satorre, E. H., & Menéndez, A. N. (2007). Sensitivity of ceres-maize simulated yields to uncertainty in soil properties and daily solar radiation. *Agricultural Systems*, *94*. doi: 10.1016/j.agsy.2006.08.003

Boyer, C. N., Larson, J. A., Roberts, R. K., McClure, A. T., Tyler, D. D., & Zhou, V. (2013). Stochastic corn yield response functions to nitrogen for corn after corn, corn after cotton, and corn after soybeans. *Journal of Agricultural and Applied Economics*, *45*. doi: 10.1017/s1074070800005198

Bullock, D. G., & Bullock, D. S. (1994). Quadratic and quadratic-plus-plateau models for predicting optimal nitrogen rate of corn: A comparison. *Agronomy Journal*, *86* . doi: 10.2134/agronj1994.00021962008600010033x

Cabas, J. H., Leiva, A. J., & Weersink, A. (2008). Modeling exit and entry of farmers in a crop insurance program. In (Vol. 37). doi: 10.1017/S1068280500002173

Cameron, T. A., & Quiggin, J. (1994). Estimation using contingent valuation data from a dichotomous choice with follow-up questionnaire. *Journal of Environmental Economics and Management*, *27* . doi: 10.1006/jeem.1994.1035

Cao, R., Carpentier, A., & Gohin, A. (2011). Measuring farmers' risk aversion: the unknown properties of the value function. *2011 International Congress, ...*

Cerrato, M. E., & Blackmer, A. M. (1990). Comparison of models for describing; corn yield response to nitrogen fertilizer. *Agronomy Journal*, *82* . doi: 10.2134/agronj1990.00021962008200010030x

Chambers, R. G., Chung, Y., & Färe, R. (1996). Benefit and distance functions. *Journal of Economic Theory*, *70* . doi: 10.1006/jeth.1996.0096

Chung, Y. H., Färe, R., & Grosskopf, S. (1997). Productivity and undesirable outputs: A directional distance function approach. *Journal of Environmental Management*, *51* . doi: 10.1006/jema.1997.0146

Coble, K. H., Knight, T. O., Patrick, G. F., & Baquet, A. E. (2002). Understanding the economic factors influencing farm policy preferences. *Review of Agricultural Economics*, *24* . doi: 10.1111/1467-9353.00021

Dalhaus, T., Barnett, B. J., & Finger, R. (2020). Behavioral weather insurance: Applying cumulative prospect theory to agricultural insurance design under narrow framing. *PLoS ONE*, *15* . doi: 10.1371/journal.pone.0232267

Dinar, A., & Yaron, D. (1992). Adoption and abandonment of irrigation technologies. *Agricultural Economics*, *6* . doi: 10.1016/0169-5150(92)90008-M

Dogan, E., Copur, O., Kahraman, A., Kirnak, H., & Guldur, M. E. (2011). Supplemental irrigation effect on canola yield components under semiarid climatic conditions. *Agricultural Water Management*, *98* . doi: 10.1016/j.agwat.2011.04.006

Dowling, J. A., Rinaldi, K. Z., Ruggles, T. H., Davis, S. J., Yuan, M., Tong, F., . . . Caldeira, K. (2020). Role of long-duration energy storage in variable renewable electricity systems. *Joule*, *4* . doi: 10.1016/j.joule.2020.07.007

Ellsberg, D. (1961). Risk, ambiguity, and the savage axioms. *Quarterly Journal of Economics*, *75* . doi: 10.2307/1884324

Eurosif. (2018, 6). *Eurosif 2018 sri study.* Retrieved from https://www.eurosif.org/wp-content/uploads/2022/06/Eurosif-Report-June-22-SFDR-Policy-Recommendations.pdf

Feder, G., Just, R. E., & Zilberman, D. (1985). Adoption of agricultural innovations in developing countries: a survey. *Economic Development Cultural Change*, *33* . doi: 10.1086/451461

Fernández, C., Koop, G., & Steel, M. F. (2002). Multiple-output production with undesirable outputs: An application to nitrogen surplus in agriculture. *Journal of the American Statistical Association*, *97* . doi: 10.1198/016214502760046989

Fersund, F. R. (2009). Good modelling of bad outputs: Pollution and multiple-output production. *International Review of Environmental and Resource Economics*, *3* . doi: 10.1561/101.00000021

Flaten, O., Lien, G., Koesling, M., Valle, P. S., & Ebbesvik, M. (2005). Comparing risk perceptions and risk management in organic and conventional dairy farming: Empirical results from norway. *Livestock Production Science*, *95* . doi: 10.1016/j.livprodsci.2004.10.014

Foster, A. D., & Rosenzweig, M. R. (1995). Learning by doing and learning from others: human capital and technical change in agriculture. *Journal of Political Economy*, *103* . doi: 10.1086/601447

Foster, A. D., & Rosenzweig, M. R. (2010). Microeconomics of technology adoption. *Annual Review of Economics*, *2* . doi: 10.1146/annurev.economics.102308.124433

Foudi, S., & Erdlenbruch, K. (2012). *The role of irrigation in farmers' risk management strategies in france* (Vol. 39). doi: 10.1093/erae/jbr024

Friedman, M., & Savage, L. J. (1948). The utility analysis of choices involving risk. *Journal of Political Economy*, *56* . doi: 10.1086/256692

Frittelli, M., & Gianin, E. R. (2002). Putting order in risk measures. *Journal of Banking and Finance*, *26* . doi: 10.1016/S0378-4266(02)00270-4

Fuentes-Arderiu, X., & Dot-Bach, D. (2009). Measurement uncertainty in manual differential leukocyte counting. *Clinical Chemistry and Laboratory Medicine*, *47* . doi: 10.1515/CCLM.2009.014

Färe, R., Grosskopf, S., Noh, D. W., & Weber, W. (2005). Characteristics of a polluting technology: Theory and practice. *Journal of Econometrics*, *126* . doi: 10.1016/j.jeconom.2004.05.010

Färe, R., Grosskopf, S., & Weber, W. L. (2006). Shadow prices and pollution costs in u.s. agriculture. *Ecological Economics*, *56* . doi: 10.1016/j.ecolecon.2004.12.022

Föllmer, H., & Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, *6* . doi: 10.1007/s007800200072

Goodwin, B. K. (1993). An empirical analysis of the demand for multiple peril crop insurance. *American Journal of Agricultural Economics*, *75* . doi: 10.2307/1242927

Han, X., Zhang, G., Xie, Y., Yin, J., Zhou, H., Yang, Y., ... Bai, W. (2019). Weather index insurance for wind energy. *Global Energy Interconnection*, *2* . doi: 10.1016/j.gloei.2020.01.008

Hasenkamp, G. (1976). A study of multiple-output production functions. klein's railroad study revisited. *Journal of Econometrics*, *4* . doi: 10.1016/0304-4076(76)90036-1

Hayhoe, K., Wuebbles, D., Easterling, D., Fahey, D., Doherty, S., Kossin, J., ... Wehner, M. (2018). Our changing climate. in impacts, risks, and adaptation in the united states: Fourth national climate assessment, volume ii. *Impacts, Risks, and Adaptation in the United States: Fourth National Climate Assessment, Volume II* , *II* .

Heath, C., & Tversky, A. (1991). Preference and belief: Ambiguity and competence in choice under uncertainty. *Journal of Risk and Uncertainty*, *4* . doi: 10.1007/BF00057884

Hertwig, R., Barron, G., Weber, E. U., & Erev, I. (2004). Decisions from experience and the effect of rare events in risky choice. *Psychological Science*, *15* . doi: 10.1111/j.0956-7976.2004.00715.x

Huettel, S. A., Stowe, C. J., Gordon, E. M., Warner, B. T., & Platt, M. L. (2006). Neural signatures of economic preferences for risk and ambiguity. *Neuron*, *49* . doi: 10.1016/j.neuron.2006.01.024

Ipcc. (2013). Working group i contribution to the ipcc fifth assessment report, climate change 2013: The physical science basis. *Ipcc*, *AR5* .

Ipcc. (2022). *Ar6 synthesis report outline: Climate change 2022.* Retrieved from https://www.ipcc.ch/site/assets/uploads/2021/12/IPCC-52_decisions-adopted-by-the-Panel.pdf

Iyer, P., Bozzola, M., Hirsch, S., Meraner, M., & Finger, R. (2020). Measuring farmer risk preferences in europe: A systematic review. *Journal of Agricultural Economics*, *71* . doi: 10.1111/1477-9552.12325

Kahn, B. E., & Sarin, R. K. (1988). Modeling ambiguity in decisions under uncertainty. *Journal of Consumer Research*, *15* . doi: 10.1086/209163

Kahneman, D., & Tversky, A. (1979). *Kahneman tversky (1979) - prospect theory - an analysis of decision under risk.pdf* (Vol. 47).

Kessler, R. (2021, 3). *Texas wind farms face billion-dollar losses from blackouts in 'illegal wealth transfer'.* Retrieved from https://www.windaction.org/posts/52234

Kilka, M., & Weber, M. (2001). What determines the shape of the probability weighting function under uncertainty? *Management Science*, *47* . doi: 10.1287/mnsc.47.12.1712.10239

Knight, F. H. (1921). *Risk uncertainty and profit knight* (Vol. 36).

Kooperman, Chen, Hoffman, Koven, Lindsay, Pritchard, . . . Randerson (2018). Forest response to rising co2 drives zonally asymmetric rainfall change over tropical land. *Nature Climate Change*. doi: https://doi.org/10.1038/s41558-018-0144-7

Koundouri, P., Nauges, C., & Tzouvelekas, V. (2006). Technology adoption under production uncertainty: Theory and application to irrigation technology. *American Journal of Agricultural Economics*, *88* . doi: 10.1111/j.1467-8276.2006.00886.x

Kumbhakar, S. C. (2002). Specification and estimation of production risk, risk preferences and technical efficiency. *American Journal of Agricultural Economics*, *84* . doi: 10.1111/1467-8276.00239

Kumbhakar, S. C., & Lovell, C. A. K. (2000). *Stochastic frontier analysis*. doi: 10.1017/cbo9781139174411

Kweilin Ellingrud, B. Q., Alex Kimura, & Ralph, J. (2022). Five steps to improve innovation in the insurance industry. *McKinsey & Co*. Retrieved from https://www.mckinsey.com/industries/financial-services/our-insights/five-steps-to-improve-innovation-in-the-insurance-industry

Laeven, R. J., & Stadje, M. (2014). Robust portfolio choice and indifference valuation. *Mathematics of Operations Research*, *39* . doi: 10.1287/moor.2014.0646

Lee, D. (2005). Agricultural sustainability and technology adoption: Issues and policies for developing countries. *American Journal of Agricultural Economics*, *87* . doi: 10.1111/j.1467-8276.2005.00826.x

Lempert, R., Popper, S., & Bankes, S. (2019). *Shaping the next one hundred years: New methods for quantitative, long-term policy analysis*. doi: 10.7249/mr1626

Levy, I., Snell, J., Nelson, A. J., Rustichini, A., & Glimcher, P. W. (2010). Neural representation of subjective value under risk and ambiguity. *Journal of Neurophysiology*, *103* . doi: 10.1152/jn.00853.2009

Link, J., Graeff, S., Batchelor, W. D., & Claupein, W. (2006). Evaluating the economic and environmental impact of environmental compensation payment policy under uniform and variable-rate nitrogen management. *Agricultural Systems, 91*. doi: 10.1016/j.agsy.2006.02.003

Llewelyn, R. V., & Featherstone, A. M. (1997). A comparison of crop production functions using simulated data for irrigated corn in western kansas. *Agricultural Systems, 54*. doi: 10.1016/S0308-521X(96)00080-7

Lyu, K., & Barré, T. J. (2017). Risk aversion in crop insurance program purchase decisions evidence from maize production areas in china. *China Agricultural Economic Review, 9*. doi: 10.1108/CAER-04-2015-0036

Maharjan, B., Das, S., Nielsen, R., & Hergert, G. W. (2021). Maize yields from manure and mineral fertilizers in the 100-year-old knorr–holden plot. *Agronomy Journal, 113*. doi: 10.1002/agj2.20713

Mahul, O. (2002). *Hedging in futures and options markets with basis risk* (Vol. 22). doi: 10.1002/fut.2207

Miao, Y., Mulla, D. J., Batchelor, W. D., Paz, J. O., Robert, P. C., & Wiebers, M. (2006). Evaluating management zone optimal nitrogen rates with a crop growth model. *Agronomy Journal, 98*. doi: 10.2134/agronj2005.0153

Murty, S., Russell, R. R., & Levkoff, S. B. (2012). On modeling pollution-generating technologies. *Journal of Environmental Economics and Management, 64*. doi: 10.1016/j.jeem.2012.02.005

of Sciences, N. A., Council, N. R., of Mathematical, A., & Sciences, P. (1979). Carbon dioxide and climate: a scientific assessment. Retrieved from https://nap.nationalacademies.org/catalog/12181/carbon-dioxide-and-climate-a-scientific-assessment

Paz, J. O., Batchelor, W. D., Babcock, B. A., Colvin, T. S., Logsdon, S. D., Kaspar, T. C., & Karlen, D. L. (1999). Model-based technique to determine variable rate nitrogen for corn. *Agricultural Systems, 61*. doi: 10.1016/S0308-521X(99)00035-9

Piet, L., & Bougherara, D. (2016, 3). The impact of farmers' risk preferences on the design of an individual yield crop insurance. *WORKING PAPER SMART, INARE UMR SMART*.

Platt, M. L., & Huettel, S. A. (2008). *Risky business: The neuroeconomics of decision making under uncertainty* (Vol. 11). doi: 10.1038/nn2062

Pope, R. D. (1982). Expected profit, price change, and risk aversion. *American Journal of Agricultural Economics, 64*. doi: 10.2307/1240655

Program, U. G. C. R. (2018). *Climate science special report: Fourth national climate assessment, volume i* (Vol. 1). doi: 10.7930/J0J964J6

Puntel, L. A., Sawyer, J. E., Barker, D. W., Dietzel, R., Poffenbarger, H., Castellano, M. J., . . . Archontoulis, S. V. (2016). Modeling long-term corn yield response to nitrogen rate and crop rotation. *Frontiers in Plant Science, 7*. doi: 10.3389/fpls.2016.01630

Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior and Organization, 3*. doi: 10.1016/0167-2681(82)90008-7

Raiffa, H. (1993). Decision analysis: introductory lectures on choices under uncertainty. 1968. *M.D. computing : computers in medical practice, 10*. doi: 10.2307/2987280

Ruszczy'ski, A. (2006, 8). *Stochastic programming.* John Wiley Sons, Inc. doi: 10.1002/0471667196.ess3225

Schahczenski, J. (2021, 9). *Crop insurance rules challenge organic and sustainable farming practices.* Retrieved from https://sustainableagriculture.net/blog/crop-insurance-rules-challenge-organic-and-sustainable-farming-practices

Schnitkey, G., Batts, R., Swanson, K., Paulson, N., & Zulauf, C. (2021). *Crop insurance tools.* Farmdoc.

Schultz, W., Preuschoff, K., Camerer, C., Hsu, M., Fiorillo, C. D., Tobler, P. N., & Bossaerts, P. (2008). *Review. explicit neural signals reflecting reward uncertainty* (Vol. 363). doi: 10.1098/rstb.2008.0152

Scofield, C. (1927). Irrigated crop rotations in western nebraska. *United States Department of Agriculture, Technical Bulletin*, *02* .

Shapiro, A., Tekaya, W., Soares, M. P., & Costa, J. P. D. (2013). Worst-case-expectation approach to optimization under uncertainty. *Operations Research*, *61* . doi: 10.1287/opre.2013.1229

Shephard, R. W. (1970). *Theory of cost and production functions*. doi: 10.2307/2230285

Sherrick, B. J., Zanini, F. C., Schnitkey, G. D., & Irwin, S. H. (2004). Crop insurance valuation under alternative yield distributions. *American Journal of Agricultural Economics*, *86* . doi: 10.1111/j.0092-5853.2004.00587.x

Smith, V. H., & Baquet, A. E. (1996). The demand for multiple peril crop insurance: Evidence from montana wheat farms. *American Journal of Agricultural Economics*, *78* . doi: 10.2307/1243790

Steiger, R., Damm, A., Prettenthaler, F., & Pröbstl-Haider, U. (2021). Climate change and winter outdoor activities in austria. *Journal of Outdoor Recreation and Tourism*, *34* . doi: 10.1016/j.jort.2020.100330

Strupczewski, G. (2019). What characterizes farmers who purchase crop insurance in poland? *Problems of Agricultural Economics*, *1* . doi: 10.30858/zer/103596

Sulewski, P., & K-loczko-Gajewska, A. (2014). Farmers' risk perception, risk aversion and strategies to cope with production risk: An empirical study from poland. *Studies in Agricultural Economics*, *116* . doi: 10.7896/j.1414

Thorp, K. R., DeJonge, K. C., Kaleita, A. L., Batchelor, W. D., & Paz, J. O. (2008). Methodology for the use of dssat models for precision agriculture decision support. *Computers and Electronics in Agriculture*, *64* . doi: 10.1016/j.compag.2008.05.022

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, *5* . doi: 10.1007/BF00122574

Ullah, R., Shivakoti, G. P., & Ali, G. (2015). Factors effecting farmers' risk attitude and risk perceptions: The case of khyber pakhtunkhwa, pakistan. *International Journal of Disaster Risk Reduction*, *13* . doi: 10.1016/j.ijdrr.2015.05.005

Vajda, S., Luce, R. D., & Raiffa, H. (1958). Games and decisions: Introduction and critical survey. *Journal of the Royal Statistical Society. Series A (General)*, *121* . doi: 10.2307/2342906

Valone.T. (2021). Linear global temperature correlation to carbon dioxide level, sea level, and innovative solutions to a projected 6°c warming by 2100. *Journal of*

Geoscience and Environment Protection. Retrieved from https://www.scirp.org/journal/paperinformation.aspx?paperid=107789

Vollmer, E., Hermann, D., & Mußhoff, O. (2017). Is the risk attitude measured with the holt and laury task reflected in farmers' production risk? *European Review of Agricultural Economics*, *44*. doi: 10.1093/erae/jbx004

Weaver, R. (1977). *The theory and measurement of provisional agricultural production decisions*.

Weaver, R. D. (1996). Prosocial behavior: Private contributions to agriculture's impact on the environment. *Land Economics*, *72*. doi: 10.2307/3146968

Weber, E. U. (1994). From subjective probabilities to decision weights: The effect of asymmetric loss functions on the evaluation of uncertain outcomes and events. *Psychological Bulletin*, *115*. doi: 10.1037//0033-2909.115.2.228

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, *55*. doi: 10.2307/1911158

Yilmaz, H., Merkez, M., & Unlu, N. (2017). An empirical analysis on the determinants of government-subsidised crop insurance purchase in grape production in turkey. *Erwerbs-Obstbau*, *59*. doi: 10.1007/s10341-016-0297-3

Figures

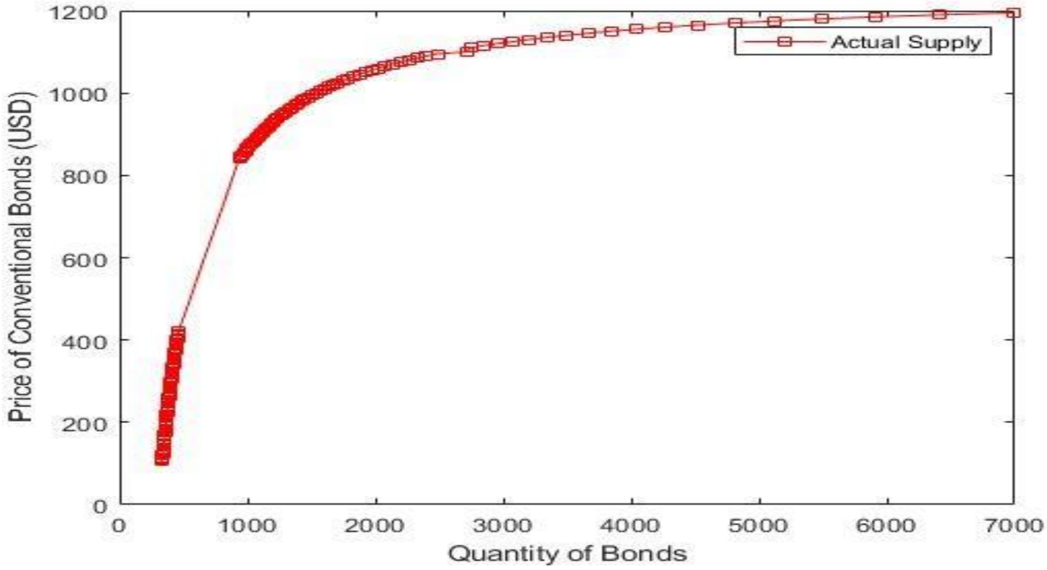

Figure 2: Conventional Bond Supply Curve in the Base Scenario

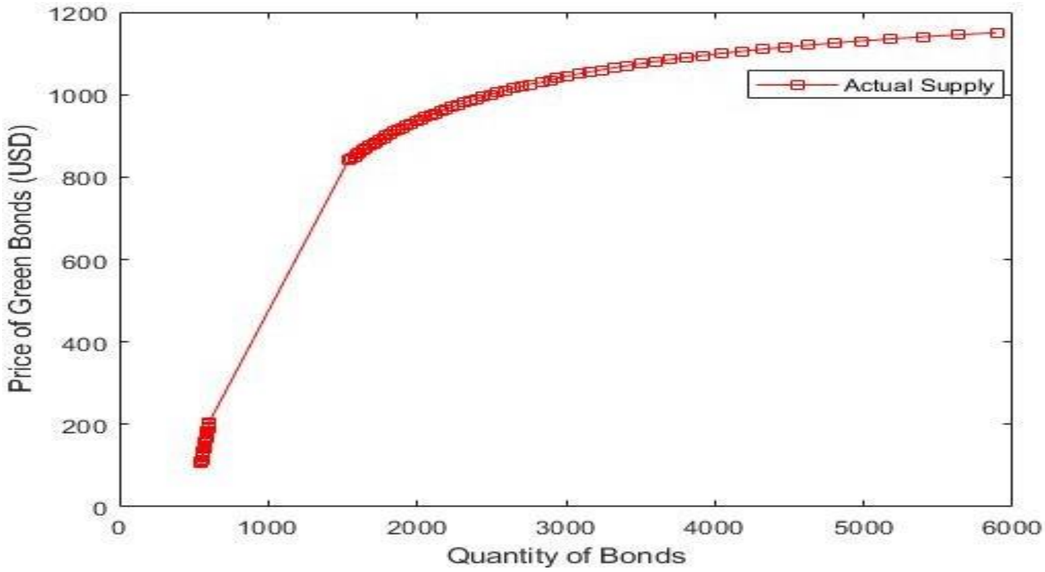

Figure 3: Green Bond Supply Curve in the Base Scenario

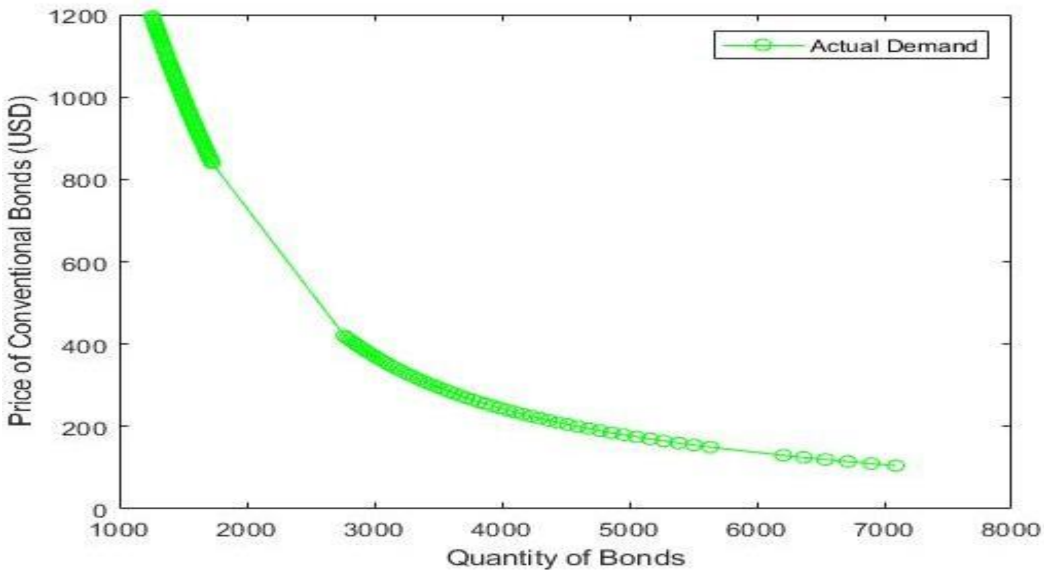

Figure 4: Conventional Bond Demand Curve in the Base Scenario

Notation: (1) Positive Δ in Exogenous Variables = Positive Change in Exogenous Variables
(2) Δ on CB Sup. = Change on Conventional Bond Supply
(3) Δ on CB Dem. = Change on Conventional Bond Supply
(4) Δ on CB Pr. = Change on Conventional Bond Price
(5) Δ on GB Sup. = Change on Green Bond Supply
(6) Δ on GB Dem. = Change on Green Bond Demand
(7) Δ on GB Pr. = Change on Green Bond Price
(8) Δ in Greenium = Price differential between CB and GB price
(9) + means positive change, - means negative change, 0 means no change, +/-/0 means either sign

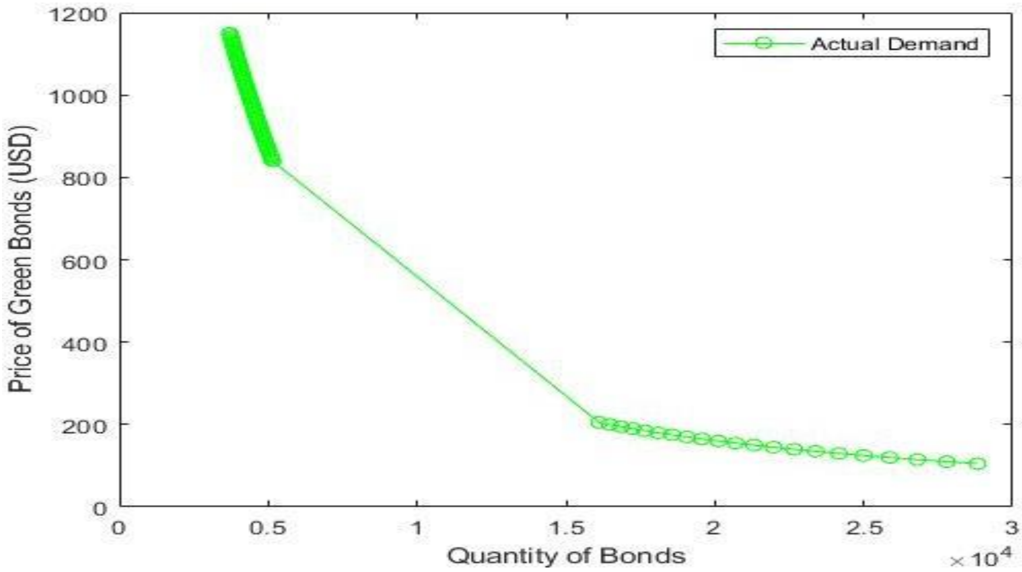

Figure 5: Green Bond Demand Curve in the Base Scenario

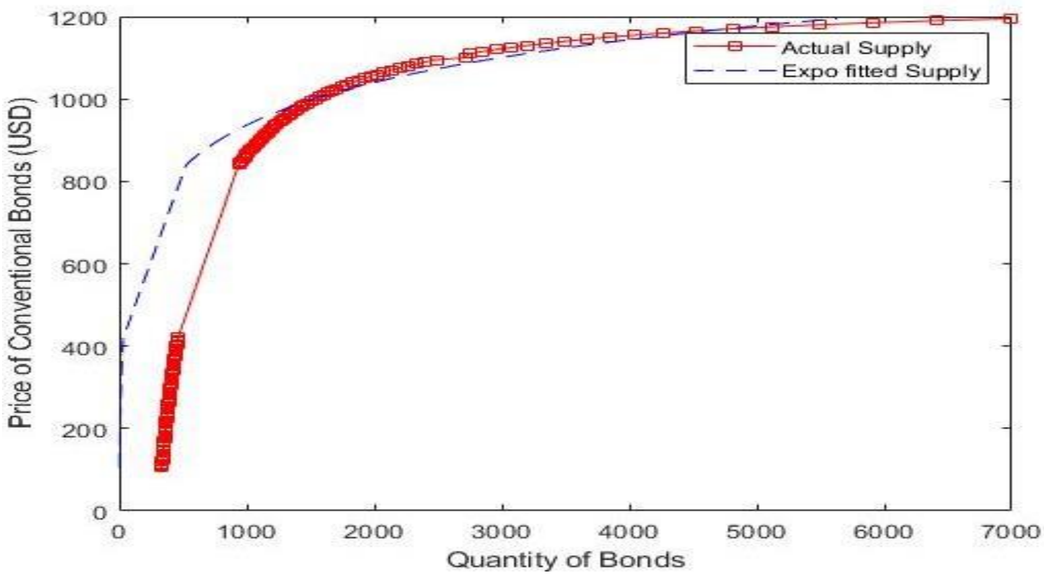

Figure 6: Actual and Fitted Conventional Bond Supply Curves (Base Scenario)

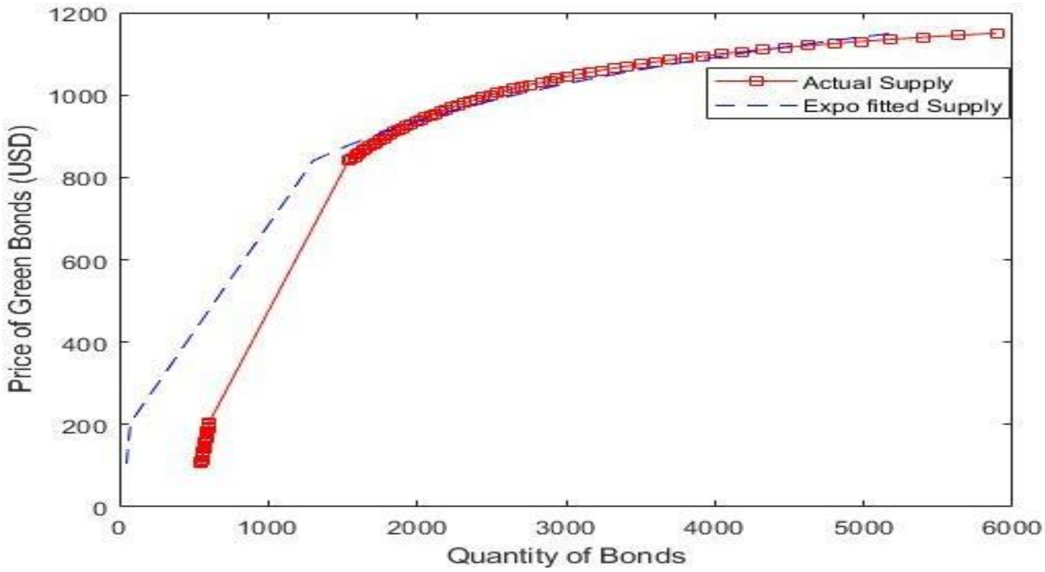

Figure 7: Actual and Fitted Green Bond Supply Curves (Base Scenario)

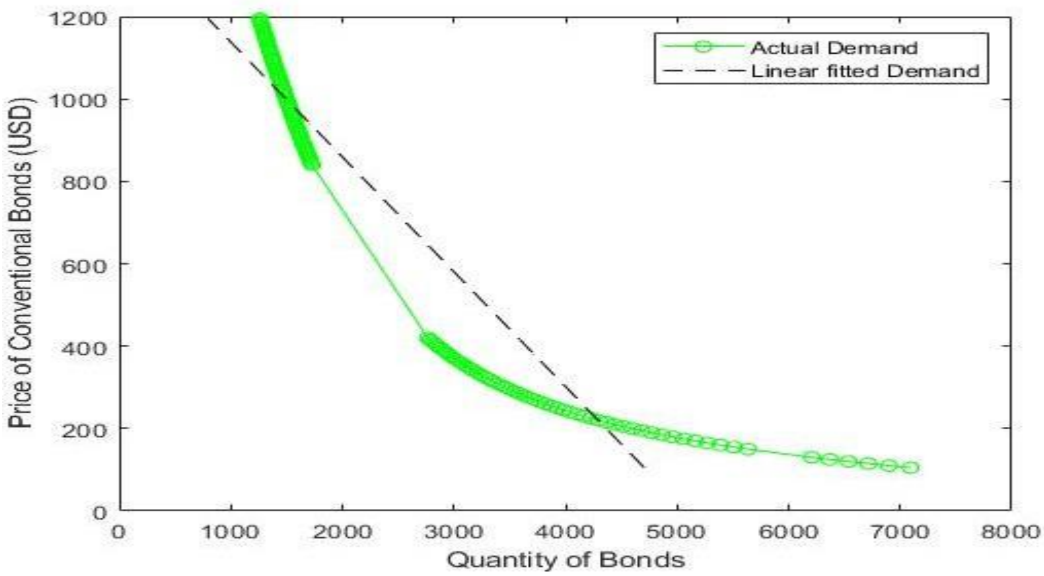

Figure 8: Actual and Fitted Conventional Bond Demand Curves (Base Scenario)

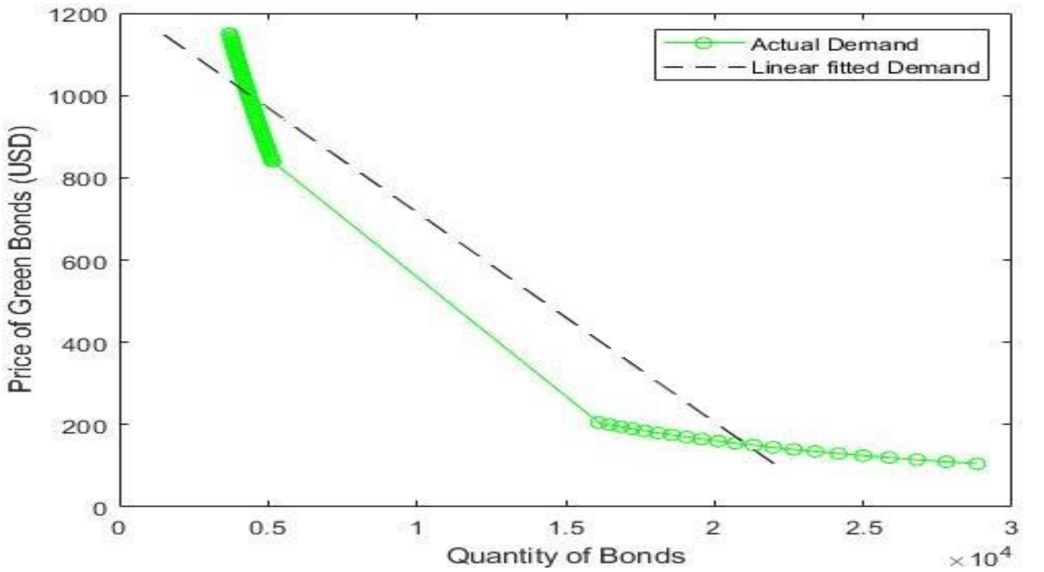

Figure 9: Actual and Fitted Green Bond Demand Curves (Base Scenario)

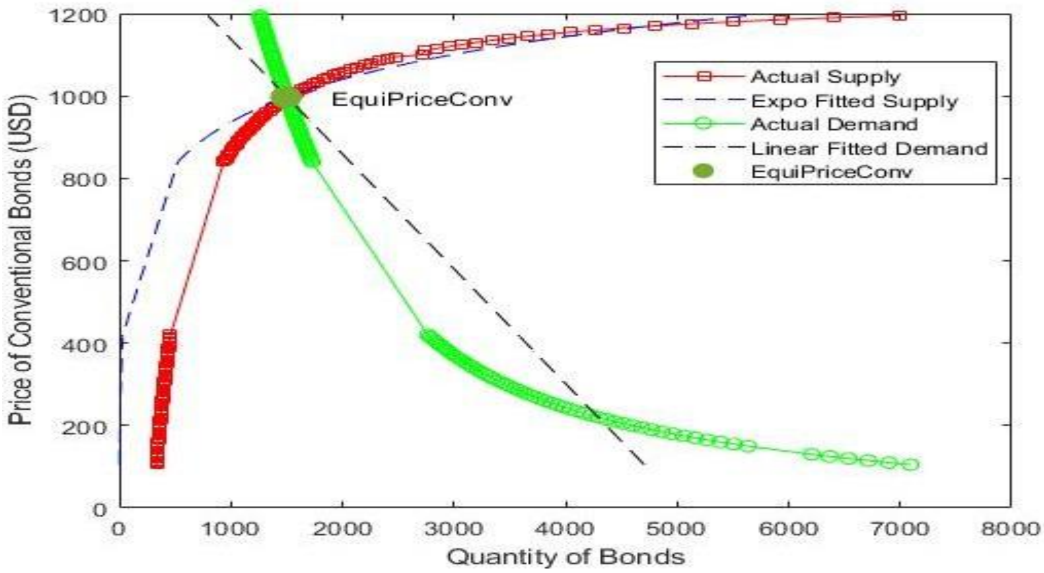

Figure 10: Equilibrium Price of Conventional Bond using Fitted Curves (Base Scenario)

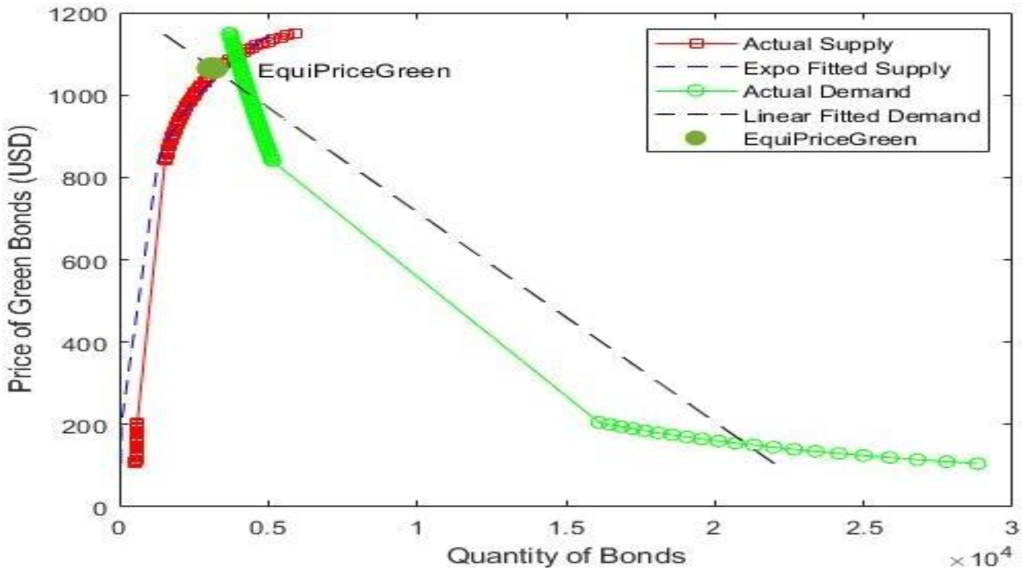

Figure 11: Equilibrium Price of Green Bond using Fitted Curves (Base Scenario)

Table 1: Hypotheses on the Direct Effect of Positive Change in Exogenous Variables on Conventional Bond Price through Demand and Supply Change

Positive Δ in Exogenous Variables	Δ *in* CB Sup.	Δ *in* CB Dem.	Δ *in* CB Pr.	Δ *in* GB Sup.	Δ *in* GB Dem	Δ *in* GB. Pr	Δ*in* Green.
Bank Interest Rate (ω)	-	-	+/-/o	-	-	+/-/o	+/-/o
Salvage Tech. Time (T_o)	+	+	+/-/o	+	+	+/-/o	+/-/o
Tech. Deteriorate. Rate (α)	-	-	+/-/o	-	-	+/-/o	+/-/o
Exp. Output Price ($E_l(P_t)$)	+	+	+/-/o	+	+	+/-/o	+/-/o
Exp. Input Price $E_l(R_t)$	+/-/o	-	+/-/o	-	-	+/-/o	+/-/o
$E(\tau_t)$: *Drought*	+	-	-	+	-	-	+/-/o
Face Value (Y)	-	-	+/-/o	-	-	+/-/o	+/-/o
Coupon Rate (c)	-	+	+	-	+	+	+/-/o
Bond Maturity (T)	+	-	-	+	-	-	+/-/o
Discount rate used by Firm (δ_l)	-	o	+	-	o	+	+/-/o
Discount Rate Investor (δ_i)	o	-	-	o	-	-	+/-/o
Bond Price (Risk-Free)	o	+	+	o	+	+	+/-/o
investor *i*ncome (ξ_i)	o	+	+	o	+	+	+/-/o
Investor Risk. Aversion Coeff. (θ_i)	o	-	-	o	-	-	+/-/o
Expected Green Effect (J)	o	o	o	+	+	+/-/o	+/-/o
Bond Price (other firms CB) (Ø= o)	o	+	+	o	o	o	-
Cost of GB transaction (C)	o	o	o	o	-	-	-
Cost of CB Issuance (I_{CB})	-	o	+	o	o	o	-
Bond Price (other firms GB) (Ø= o)	o	o	o	o	+	+	+
Cost of CB transaction (C)	o	-	-	o	o	o	+
Cost of GB Issuance (I_{GB})	o	o	o	-	o	+	+
Prosocial Behavior (η_i)	o	-	-	o	+	+	+
Environmental Target ($\overline{S}_{l,T}$)	o	o	o	-	+	+	+

Table 2: Summary Statistics of the Empiral Variables used in the Estimation of the Yield Function

Statistic	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
N (kg/ha)	207	88.9	32.1	159.95	235.38	1,294
TP_g(∘C)	22.4	1.65	15.80	21.37	23.57	27.5
W_g(*cm*)	9.97	3.07	1.9	7.74	11.89	26.9
TP_p(∘C)	13.54	2.02	6.4	12.15	14.95	19.2
W_p(*cm*)	10.6	4.11	1.94	7.71	12.59	35.24
TP_h(∘C)	15.05	1.65	8.95	13.95	16.15	20.5
W_h(*cm*)	8.48	3.84	0.982	5.6	10.75	33.1
S (gCO_2e/kg)	370	381.52	0.530	96.68	503.81	3,275

Table 3: Production Function with Direct Control of S (Green Bond Issuer)

	Dependent variable
	yield
N	4.273*
	(2.255)
TP_g	1,531.515***
	(195.027)
W_g	332.174***
	(63.434)
S	6.816***
	(0.574)
N^2	−0.005***
	(0.0004)
TP_g^2	−39.650***
	(4.253)
W_g^2	−19.516***
	(0.880)
S^2	−0.001***
	(0.00004)
$TP_g * N$	0.060
	(0.094)
$W_g * N$	−0.135***
	(0.045)
$TP_g * W_g$	11.318***
	(2.510)
N*S	0.002**
	(0.001)
$TP_g * S$	−0.079***
	(0.023)
$W_g * S$	−0.089***
	(0.010)
TP_p	−25.720
	(76.212)
TP_p^2	4.465
	(2.763)
TP_h	282.066***
	(108.840)
TP_h^2	−7.349**
	(3.461)
W_p	46.844***
	(11.879)
W_p^2	−1.922***
	(0.438)
W_h	15.180
	(11.272)
W_h^2	−1.270**
	(0.502)
Constant	−15,695.510***
	(1,922.777)
Observations	8,516
R^2	0.758
Adjusted R^2	0.757
Residual Std. Error	941.236 (df = 8468)
Note	*p<0.1; **p<0.05; ***p<0.01

The regressions contain fixed effects at the Year and County Level, and the standard deviations are clustered at the county level

Table 4: Production Function with No Control of S (Conventional Bond Issuer)

	Dependent variable
	yield
N	16.614***
	(2.895)
TP_g	3,421.894***
	(250.659)
W_g	422.343***
	(82.138)
N^2	−0.004***
	(0.001)
TP_g^2	−64.975***
	(5.447)
W_g^2	−21.545***
	(1.146)
$TP_g * N$	−0.606***
	(0.119)
$W_g * N$	−0.193***
	(0.058)
$TP_g * W_g$	9.495***
	(3.193)
TP_p	187.639*
	(100.118)
TP_p^2	−6.135*
	(3.627)
TP_h	237.291*
	(142.777)
TP_h^2	−18.386***
	(4.540)
W_p	35.701**
	(15.549)
W_p^2	−2.613***
	(0.573)
W_h	−13.135
	(14.807)
W_h^2	−1.871***
	(0.660)
Constant	−42,002.550***
	(2,448.982)
Observations	8,516
R^2	0.579
Adjusted R^2	0.577
Residual Std. Error	1,241.250 (df = 8475)
Note	*p<0.1; **p<0.05; ***p<0.01

The regressions contain fixed effects at the Year and County Level, and the standard deviations are clustered at the county level

Table 5: Simulation Parameter Values (Base Scenario)

Simulation Variables	**Values**
c_0, c_1	0.05
c_{gov}	0.02
Y_0, Y_1, Y_{gov}	1000 USD
T	5 years
T_0	7 years
b_1, b_{gov}	1000 USD
ξ_i	$\log N(\mu_{\xi_i} = log(35384), \sigma_{\xi_i} = log(5000))$
β_i	$N(\mu_{\beta_i} = 0.5, \sigma_{\beta_i} = 0.08)$
η_i	$N(\mu_{\eta_i} = 0.5, \sigma_{\eta_i} = 0.08)$
δ	0.07
ρ	0.75
$\hat{S}^*_{I_0,0}$	20000
P	0.23 $
R_W	0.02 $ per m^3
R_N	1.3 $ per kg
κ_W	10^-2 cm per deg CO2eq
κ_{TP}	10^-2 per CO2eq
$\phi_{Rg\,min}$	$600 W m^2$
σ_{SB}	$5.67.^* 10^-8$
ζ	1-5
α_{IC}	0.01
w	0.06

Table 6: Statistics on the Base Scenario

Statistics	Base Scenario
Avg_demand_conv	2676.16
Avg_demand_green	8629.58
Avg_supply_conv	1362.74
Avg_supply_green	2254.46
$Equi\_\widehat{price}\_conv$	997.54
$Equi\_\widehat{price}\_green$	1052.82
$Env\_\widehat{Default}$	1150.02
$\widehat{Greenium}$	55.28

Table 7: Magnitude of the Change in the Greenium for a Declining Rate in Precipitation during Growing Time

Statistics	Case 1	Case 2	Case 3	Case 4	Case 5
Decrease in precip. rate (cm)	1.25	2.5	3.75	5	6.75
Avg_demand_conv (units)	2676.16	2600.02	2413.06	2240.53	2100.53
Avg_demand_green (units)	8629.59	7409.37	6742.84	6026.93	5501.13
Avg_supply_conv (units)	1362.74	1355.25	1364.41	1364.18	1363.20
Avg_supply_green (units)	2254.46	2297.79	2254.42	2300.59	2300.46
Equi_price_conv (USD)	997.55	996.19	996.31	995.76	995.24
Equi_price_green (USD)	1052.83	1043.35	1045.91	1050.34	1053.56
Env_Default (CO2e)	1150.03	1150.18	1150.14	1150.06	1150.003
Greenium (USD)	55.28	47.16	49.6	54.58	58.32

Table 8: Elasticities on the Change in the Greenium for a Declining Rate in Precipitation during Growing Time

Statistics	Case 2	Case 3	Case 4	Case5
Decrease in precip. rate (cm)	2.5	3.75	5	6.75
Avg_demand_conv (%)	-2.85	-9.83	-16.28	-21.51
Avg_demand_green (%)	-14.14	-21.86	-30.16	-36.25
Avg_supply_conv (%)	-0.55	0.12	0.11	0.03
Avg_supply_green (%)	1.92	-0	2.05	2.04
Equi_price_conv (%)	-0.14	-0.12	-0.18	-0.23
Equi_price_green (%)	-0.9	-0.66	-0.24	0.07
Env_Default (%)	0.01	0.01	0	-0
Greenium (%)	-14.69	-10.27	-1.27	5.5

Table 9: Magnitude of the Change in the Greenium for an Increasing Rate in Temperature during Growing Time

Statistics	Case 1	Case 2	Case 3	Case 4	Case 5
Increase in Temperature ($\circ C$)	1.25	2.5	3.75	5	6.75
Avg_demand_conv	2676.16	2475.5	1984.27	762.88	762.88
Avg_demand_green	8629.59	6140.99	4969.68	4886.17	4886.17
Avg_supply_conv	1362.74	838.51	1292.27	1438.63	1438.62
Avg_supply_green	2254.46	2286.02	2123.14	1233.61	1233.60
Equi_price_conv	997.55	948.24	983.11	621.26	621.26
Equi_price_green	1052.83	1044.93	1045.13	1112.57	1112.57
Env_Default	1150.03	1194.19	1274.06	1346.82	1346.82
Greenium	55.28	96.7	62.02	491.31	491.31

Table 10: Elasticities on The Change in the Greenium for an Increasing Rate in Temperature during Growing Time

Statistics	Case2	Case 3	Case 4	Case 5
Increase in Temperature ($\circ C$)	2.5	3.75	5	6.75
Avg_demand_conv (%)	-7.5	-25.85	-128.51	-128.51
Avg_demand_green (%)	-28.84	-42.41	-43.38	-43.38
Avg_supply_conv (%)	-38.47	-5.17	5.57	5.57
Avg_supply_green (%)	1.4	-5.82	-45.28	-45.28
Equi_price_conv (%)	-4.94	-1.45	-37.72	-37.72
Equi_price_green (%)	-0.75	-0.73	5.67	5.67
Env_Default (%)	3.84	10.78	17.11	17.11
Greenium (%)	74.93	12.19	788.77	788.77

Table 11: Magnitude of the Change in the Greenium for an Increasing Nitrogen Input Price

Statistics	Case 1	Case 2	Case 3	Case 4	Case 5
Nitrogen Price	1.3	2.6	3.9	5.2	6.5
Avg_demand_conv	2676.16	1977.64	2223.87	2256.34	1533.21
Avg_demand_green	8629.59	24695.54	30642.1	34016.89	34372.47
Avg_supply_conv	1362.74	2229.15	1618.56	1512.48	5173.57
Avg_supply_green	2254.46	2439.89	2356.04	2179.31	1949.57
Equi_price_conv	997.55	903.98	610.19	315.16	170.37
Equi_price_green	1052.83	1270.81	1380.61	1385.19	1403.83
Env_Default	1150.03	-1784.91	-4717.56	-7649.29	-10580.80
Greenium	55.28	366.83	770.42	1070.03	1233.46

Table 12: Elasticities on The Change in the Greenium for an increasing Nitrogen Input Prices

Statistics	Case 2	Case 3	Case 4	Case 5
Nitrogen Price	2.6	3.9	5.2	6.5
Avg_demand_conv (%)	-26.1	-16.9	-15.69	-42.71
Avg_demand_green (%)	186.17	255.08	294.19	298.31
Avg_supply_conv (%)	63.58	18.77	10.99	279.65
Avg_supply_green (%)	8.23	4.51	-3.33	-13.52
Equi_price_conv (%)	-9.38	-38.83	-68.41	-82.92
Equi_price_green (%)	20.7	31.13	31.57	33.34
Env_Default (%)	-255.21	-510.21	-765.14	-1020.05
Greenium (%)	563.59	1293.67	1835.65	2131.3

Table 13: Magnitude of the Change in the Greenium for Increasing Prosocial Preferences within the Population (Under High Environmental Promise)

Statistics	Case 1	Case 2	Case 3	Case 4	Case 5
Prosocial Preference	0.5	0.6	0.7	0.8	0.9
Avg_demand_conv	2676.16	2676.16	2676.16	2676.16	2676.16
Avg_demand_green	8629.59	8629.53	8621.91	8367.28	8367.28
Avg_supply_conv	1362.74	1362.74	1362.74	1362.74	1362.74
Avg_supply_green	2254.46	2254.46	2254.46	2254.46	2254.46
Equi_price_conv	997.55	997.55	997.55	997.55	997.54
Equi_price_green	1052.83	1052.83	1052.74	1049.82	1049.82
Env_Default	1150.03	1150.03	1150.03	1150.07	1150.07
Greenium	55.28	55.28	55.2	52.27	52.27

Table 14: Elasticities on the Change in the Greenium for Increasing Prosocial Preferences within the Population (Under High Environmental Promise)

Statistics	Case 2	Case 3	Case 4	Case 5
Prosocial Preference	0.6	0.7	0.8	0.9
Avg_demand_conv (%)	0	0	0	0
Avg_demand_green (%)	-0	-0.09	-3.04	-3.04
Avg_supply_conv (%)	0	0	0	0
Avg_supply_green (%)	0	0	0	0
Equi_price_conv (%)	0	0	0	-0
Equi_price_green (%)	0	-0.01	-0.29	-0.29
Env_Default (%)	0	0	0	0
Greenium (%)	0	-0.14	-5.45	-5.46

Table 15: Magnitude of the Change in the Greenium for Higher Solar Efficiency

Statistics	Case 1	Case 2	Case 3	Case 4	Case 5
Minimum Solar radiation	600	400	300	200	100
Avg_demand_conv	2676.16	2676.03	2676.16	2666.31	2686.08
Avg_demand_green	8629.59	8570.66	8821.36	8821.36	8821.36
Avg_supply_conv	1362.74	1366.08	1364.28	1362.56	1354.98
Avg_supply_green	2254.46	2298.19	2254.15	2254.57	2254.52
Equi_price_conv	997.55	997.26	997.25	996.59	996.55
Equi_price_green	1052.83	1054.14	1054.73	1054.71	1054.71
Env_Default	1150.03	1150	1112.82	1112.82	1112.82
Greenium	55.28	56.88	57.48	58.12	58.15

Table 16: Elasticities on The Change in the Greenium for Higher Solar Efficiency

Statistics	Case 2	Case 3	Case 4	Case 5
Minimum solar radiation	400	300	200	100
Avg_demand_conv (%)	-0	0	-0.37	0.37
Avg_demand_green (%)	-0.68	2.22	2.22	2.22
Avg_supply_conv (%)	0.25	0.11	0.87	-0.57
Avg_supply_green (%)	1.94	-0.01	0	0
Equi_price_conv (%)	-0.03	-0.03	0.04	-0.1
Equi_price_green (%)	0.12	0.18	0.18	0.18
Env_Default (%)	-0	-3.24	-3.24	-3.24
Greenium (%)	2.89	3.98	5.13	5.21

Table 17: Magnitude of the Change in the Greenium for Increasing Green Bond Issuance Cost

Statistics	Case 1	Case 2	Case 3	Case 4	Case 5
Cost of green bond issuance	0.01	0.02	0.04	0.06	0.08
Avg_demand_conv	2676.16	2676.16	2676.16	2676.16	2676.16
Avg_demand_green	8629.59	8171.07	8255.68	8425.23	8092.80
Avg_supply_conv	1362.74	1362.74	1362.74	1362.74	1362.74
Avg_supply_green	2254.46	2101.65	1953.24	1660.95	1677.43
Equi_price_conv	997.55	997.55	997.55	997.55	997.55
Equi_price_green	1052.83	1065.95	1097.52	1120.94	1139.52
Env_Default	1150.03	1150.19	1150.24	1150.41	1150.48
Greenium	55.28	68.4	99.97	123.4	141.97

Table 18: Elasticities on the Change in the Greenium for Increasing Green Bond Issuance Cost

Statistics	Case 2	Case 3	Case 4	Case 5
Cost of green bond issuance	0.02	0.04	0.06	0.08
Avg_demand_conv (%)	0	0	0	0
Avg_demand_green (%)	-5.31	-4.33	-2.37	-6.22
Avg_supply_conv (%)	0	0	0	0
Avg_supply_green (%)	-6.78	-13.36	-26.33	-25.6
Equi_price_conv (%)	0	0	0	-0
Equi_price_green (%)	1.25	4.24	6.47	8.23
Env_Default (%)	0.01	0.02	0.03	0.04
Greenium (%)	23.73	80.84	123.23	156.83