Some theorems on the class $M$-$A(n^*)$ operators on Hilbert space

Shaymaa Shawkat Al-shakarchi

Department of Mathematics, Faculty of Basic Education
University of Kufa, Najaf, Iraq

Abstract:

An operator $T_1 \in B(H)$ is referred to as $M - A(n^*)$ operators if $(T_1^*[|T_1|^{2n}T_1])^{1/2} - M|T_1^*|^2 \geq 0$ for a positive integer $n$. The well-known Fuglede–Putnam’s theorem states that the operator equation $T_1X = XT_2$ implies $T_1^*X = XT_2^*$ when $T_1$ and $T_2$ are normal operators. This work demonstrates that if $X$ is a Hilbert–Schmidt operator, $T_1$ belongs to the class $M - A(n^*)$ operators and $T_2^*$ is an invertible operator belonging to the class $M - A(n^*)$ operators such that $T_1X = XT_2$, then $T_1^*X = XT_2^*$.

Keywords: Hilbert space, Fuglede–Putman, $A(n^*)$ operators,

1- Introduction:

Let $H$ denote a separable complex Hilbert space characterized by the inner product $<;>$. The space denoted as $B(H)$ represents the set of all bounded linear operators on $H$, whereas $I = IH$ represents the identity operator.

An operator $T \in B(H)$ is said to be positive (denoted $T \geq 0$) if $<Tx, xi> \geq 0$ for all $x \in H$. The null operator and the identity on $H$ will be denoted by $O$ and $I$, respectively. If $T$ is an operator, then $T^*$ is its adjoint, and $||T|| = ||T^*||$. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number $\lambda$ by $\bar{\lambda}$. The closure of a set $M$ will be denoted by $\bar{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. We write $\sigma(T), \sigma_p(T)$ and $\sigma_a(T)$ for the spectrum, point spectrum and approximate point spectrum, respectively. Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by $iso\sigma(T)$ and $acc\sigma(T)$, respectively.

For an operator $T$, as usual, $|T| = (T^*T)^{1/2}$ and $[T^*, T] = T^*T - TT^*$ (the self–commutator of $T$). In the following we will mention some known classes of operators defined in Hilbert space $H$. An operator $T$, is said to be normal, if $[T^*, T] = 0$, and $T$ is said to be hyponormal, if $[T^*, T]$ is nonnegative, equivalently if $|T|^2 \geq |T^*|^2$. An operator $T$ is a paranormal if $||T^*x|| \geq ||T x||^2$, and it is $M$–paranormal if $M ||T^2x|| \geq ||T x||^2$ (see [3]) for every unit vector $x \in H$. In [5] authors, Furuta, Ito and Yamazaki introduced the A class of operators, respectively class $A(n)$ of operators defined as follows: for each $n > 0$, an operator $T$ is a class $A(n)$ operator if

$$ (T^*|T|^{2n}T)^{1/2n+1} \geq |T|^2, $$

(for $n = 1$ it defines the class $A$ operators) which includes the class of log-hyponormal operators (see Theorem 2, in [5]) and it is included in the class of paranormal operators, in case where $n = 1$ (see Theorem 1 in [5]). In the same paper the absolute-$n$-paranormal operators were introduced as follows: For each $n > 0$, an operator $T$ is an absolute-$n$-paranormal operator if

$$ |||T|^nT x|| \geq ||T x||^{n+1}, $$

for every $n > 0$. In case where $n = 1$ it defines the class $A^*$ operators. Every class $A^*$ operator is a $*-$paranormal operator, Theorem 1.3 in [4]. In paper [10] the absolute-$n$* -$paranormal class of operators was introduced as follows:
\[\|T\|^n T x\| \geq \|T^* x\|^{n+1}.\]

For each \(n > 0\), every class \(A(n^*)\) operator is an absolute-\(n^*\) paranormal operator, Theorem 2.4 in [10].

**Definition 1-1**

For each \(n > 0, M > 0\) an operator \(T\) is a class \(M - A(n^*)\) operator if \((T^*|T|^{2n+1})^{\frac{1}{n+1}} \geq M |T^*|^2\).

Any an absolute-\(n^* - M\)-paranormal operator, if for each \(n > 0, M > 0, \|T\|^n T x\| \geq M \|T^* x\|^{n+1}\), for every unit vector \(x \in H\) and every class \(A(n^*)\) is an absolute-\(n^* - M\)-paranormal operator [10].

The primary objective of this work is to demonstrate that a-Browder's and Fuglede-putnam theorems are applicable to \(M - A(n^*)\) operator.

**2. Browder's theorem**

If there is a vector \(x \neq 0\) that satisfies \((T - \lambda)x = 0\), then a complex number \(\lambda \in \mathbb{C}\) is said to be in the point spectrum \(\sigma_p(T)\) of the operator \(T\). If \(T \in B(H)\), we may refer to the null space as \(N(T)\) and the range as \(R(T)\).

The spectrum and the approximate point spectra of \(T\) are indicated as \(\sigma(T)\) and \(\sigma_a(T)\) respectively. The Fredholm operator \(T\) is defined as follows: \(R(T)\) is closed, \(\alpha(T) = \dim N(T) < \infty\), and \(\beta(T) = \dim H/R(T) < \infty\). Furthermore, if the \(\text{ind}(T) = \alpha(T) - \beta(T) = 0\), then \(T\) is known as the Weyl operator. The essential spectrum, denoted as \(\sigma_e(T)\), and the Weyl spectrum, denoted as \(\sigma_w(T)\), are theoretically defined as follows:

\[\sigma_e(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not Fredholm}\},\]

and

\[\sigma_w(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not Weyl}\}.\]

An operator \(T \in B(H)\) has the finite ascent if \(N(T^m) = N(T^{m+1})\) for a positive integer \(m\), and finite descent if \(R(T^n) = R(T^{n+1})\) for a positive integer \(n\). If the operator \(T\) is Fredholm of finite ascent and descent, it is referred to be Browder. The Browder spectrum of \(T\) may be expressed as:

\[\sigma_b(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not a Browder compound}\}.\]

The Browder's theorem applies to \(T\) if \(\sigma_w(T) = \sigma_b(T)\).

The operator \(T\) has the single valued extension property, which is known as SVEP. This property is defined as follows: if \(f(z)\) is an analytic vector valued function on an open set \(D \subset \mathbb{C}\), such that \((T - \lambda)f(z) = 0\) for all \(z \in D\), then \(f(z) = 0\) for all \(z \in D\).

This section demonstrates that a-Browder’s theorem applies to the class \(M - A(n^*)\) operators.

**Theorem 2.2**

Let \(T \in B(H)\) be an operator in the class \(M - A(n^*)\). If \((T - \lambda)x = 0\), then \((T^* - \bar{\lambda})x = 0\) for all \(\lambda \in \mathbb{C}\).

**Proof:**

Since \(\langle M|T^*|^2 x, x \rangle \leq \langle (T^*|T|^{2n+1})^{\frac{1}{n+1}} x, x \rangle = \langle T^* T x, x \rangle = |\lambda|^2 \|x\|^2\).

Thus, \(\|T^* x - \bar{\lambda} x\|^2 = \langle T^* x - \bar{\lambda} x, T^* x - \bar{\lambda} x \rangle = \langle T^* x, T^* x \rangle - \bar{\lambda} \langle T^* x, x \rangle - \bar{\lambda} \langle x, T^* x \rangle + |\lambda|^2\)

\[= |\langle T^* x, x \rangle| - \bar{\lambda} \langle T x, x \rangle - \bar{\lambda} \langle x, T x \rangle + |\lambda|^2\]
\[
\leq |\lambda|^2 - |\lambda|^2 |\lambda|^2 + |\lambda|^2 = 0.
\]

Hence, \(T^*x = \overline{\lambda}x\).

**Lemma 2.3**

If \(T\) belongs to the class \(M - A(n^*)\), then \(T - \lambda\) has finite ascent for each \(\lambda \in \mathbb{C}\).

**Proof:**

Since \(T\) is a class \(M - A(n^*)\) operator, it follows that \(N(T - \lambda) \subseteq N(T^* - \lambda)\), for each \(\lambda \in \mathbb{C}\) by Theorem 2.2. Therefore, it is possible to express \(T - \lambda\) as the following 2x2 operator matrix in relation to the decomposition \(N(T - \lambda) \oplus N(T^* - \lambda)\):

\[
T - \lambda = \begin{bmatrix} 0 & 0 \\ 0 & T_1 \end{bmatrix}.
\]

Let \(x \in N((T - \lambda)^2)\), and let’s write \(x = a + b\), where \(a \in N(T - \lambda)\) and \(b \in N(T - \lambda)^\perp\). Then \(0 = (T - \lambda)^2 x = (T - \lambda)^2 b\), so that \((T - \lambda)b \in N(T - \lambda) \cap N(T - \lambda)^\perp = \{0\}\), which implies that \(b \in N(T - \lambda)\), and hence \(x \in N(T - \lambda)\). Therefore \(N(T - \lambda) = N(T - \lambda)^2\).

**Corollary 2.4**

If \(T \in\) the class \(M - A(n^*)\) operator, then \(T\) possess the property of SVEP.

**Proof:**

The Proof may be derived straight from Lemma 2.3 and Proposition 1.8 as presented in reference [8].

In this demonstration, we shall establish the validity of a-Browder's theorem for the class M-A(n*) operator. To do this, we require the subsequent definitions.

**Definition 2.5**

The Browder essential approximate point spectrum \(\sigma_{ab}(T)\) of \(T\) is defined by

\[
\sigma_{ab}(T) = \cap \{\sigma_a(T + K) : TK = KT, K \text{ is a compact operator}\}.
\]

**Definition 2.6**

We say that \(a\)-Browder’s theorem holds for \(T\) if \(\sigma_{ea}(T) = \sigma_{ba}(T)\). It is well known that \(a\)-Browder’s theorem \(\Rightarrow\) Browder’s theorem.

**Theorem 2.7**

Let \(T \in B(H)\) be a class \(M - A(n^*)\) operator. Then \(T\) obeys \(a\)-Browder’s theorem.

**Proof:**

The SVEP of any operator in the class M-A(n*) operator implies that \(T\) satisfies a-Browder's theorem, as stated in Theorem 2.8 in reference [11].

3- Fuglede-Putnam theorem

The use of the Fuglede-Putnam theorem has significant importance within the realm of products, encompassing sums, which are composed of normal operators. An instance illustrating the use of this theory is the Kaplansky theorem [6]. Many scholars in the field of mathematics strive to further develop this theorem within the framework of nonnormal operators (see to [11]).

This is the well-known Fuglede-Putnam theorem, as stated in reference [3].

**Theorem 3-1**

Consider two operators \(T_1\) and \(T_2\) be normal operator and \(X\) be an operator such that \(T_1X = XT_2\), then \(T_1^*X = XT_2^*\).
Assume $T$ is an operator in the space $B(H)$ and $\{e_i\}$ be an orthonormal basis for $H$. We define the Hilbert-Schmidt norm as $\|T\|_2 = (\sum_{i=1}^{\infty} \|Te_i\|^2)^{\frac{1}{2}}$. This definition is independent of the choice of basis (see [2]). If $\|T\|_2 < \infty$, then $T$ is said to be a Hilbert-Schmidt operator and we denote the Hilbert-Schmidt class by $C_2(H)$. The set $C_2(H)$ form an ideal of the algebra $B(H)$. The ideal is a Hilbert space with an inner product $(X_1, X_2) = \sum_{i=1}^{\infty} (X_1e_i, X_2e_i) = tr(X_1^*X_2) = tr(X_1^*X_2^*)$. For each pair of operators $T_1, T_2 \in B(H)$, there is an operator $T_{T_1T_2}$ defined on $C_2(H)$ via the formula $T_{T_1T_2}(X) = T_1XT_2$ in [3]. Obviously $\|T\| \leq \|T_1\| \|T_2\|$. The adjoint of $\Gamma$ is obtained by the formula $\Gamma^{*T_1T_2}X = T_1^*XT_2^*$, as stated in the reference [1].

**Theorem 2.1.** If $T$ is an invertible $M - A(n^*)$ operator for $M > 0$, then $T^{-1}$ is also be an absolute $M - A(n^*)$ operator.

**Proof:**
Given that $(T^*|T|^{2nT})^{\frac{1}{n+1}} = (T^{(n+1)}T^{(n+1)}T^{(n+1)})^{\frac{1}{n+1}}$, it follows that $(TT^*)(n+1) \leq (T^*T)(n+1)$. 

Thus, $(T^*T)\frac{-(n+1)}{2}((TT^*)(n+1)(TT^*)^{-\frac{(n+1)}{2}} \leq I$ and $(T^*T)\frac{(n+1)}{2}(TT^*)^{-\frac{(n+1)}{2}} \geq I$. 

This is equivalent to $(TT^*)^{-\frac{(n+1)}{2}} - (T^*T)^{-\frac{(n+1)}{2}} = (T^{-1}T^{-1})^{(n+1)} - (T^{-1}T^{-1})^{(n+1)} \geq 0$. 

The class $M - A(n^*)$ operator includes the operator $T^{-1}$.

**Theorem 3-2**
If $T_1, T_2$ and $T_2^*$ are $M - A(n^*)$ operators, then the operator $\Gamma_{T_1T_2}$ belongs to $M - A(n^*)$ operators’ class.

**Proof:**
Since $\Gamma_{T_1T_2} \Gamma_{T_1T_2}X = T_1^*T_2^*T_2^*T_1^*X$ and $\Gamma_{T_1T_2} \Gamma_{T_1T_2}X = T_1^*T_2^*T_2^*T_1^*X$ for any operator $X$ in $C_2(H)$. 

We get $\|\Gamma_{T_1T_2}X = \|T_1X|T_2^*|X = \|T_1|X|T_2^*|. \|\Gamma_{T_1T_2}X = \|\Gamma_{T_1T_2}X = \|T_1|X|T_2^*|X = \|T_1|X|T_2^*|. \|

As well as, $\|\Gamma_{T_1T_2}X = \|T_1^2X|T_2^*|^2$ and $\|\Gamma_{T_1T_2}X = \|\Gamma_{T_1T_2}X = \|\Gamma_{T_1T_2}X = \|T_1^2X|T_2^*|^2$. 

We have $\|\Gamma_{T_1T_2}X = \|T_1^2X|T_2^*|^2$ and $\|\Gamma_{T_1T_2}X = \|T_1^2X|T_2^*|^2$ for each $n > 0$. 

Thus, $\|\Gamma_{T_1T_2}X = \|\Gamma_{T_1T_2}^{2n}X = \|T_1^2X|T_2^*|^2 \geq (|T_1|^2)^{n+1}X$ 

$\geq (\|T_2^*|^2)^{n+1}X$. 

**Theorem 3-3**
Let $T_1$ and $T_2$ be $M - A(n^*)$ operator such that $T_2^*$ is invertible operator in the class of $M - A(n^*)$ operators, and let $X$ be a Hilbert-Schmidt operator. If $T_1X = XT_2$, then $T_1^*X = XT_2^*$. 

**Proof:**
Let $\Gamma_{T_1T_2}$ be the Hilbert-Schmidt operator defined by $\Gamma_{T_1T_2}X = T_1X(T_2)^{-1}$. 

Since $T_1, T_2$ are in the class $M - A(n^*)$ operators, by Theorem 3-2, $\Gamma_{T_1T_2}$ is $M - A(n^*)$ operator. The hypothesis $T_1X = XT_2$, implies that $T_1X(T_2)^{-1} = X$. 

And $\Gamma_{T_1T_2}X = X$, $\Gamma_{T_1T_2}X = X$ by Theorem. 

Hence, we have $T_1^*X(T_2^{-1})^* = X$. Therefore, $T_1^*X = XT_2^*$. 

Shaymaa Shawkat Al-shakarchi, IJSRM Volume 12 Issue 04 April 2024 M-2024-482
4-Conclusion
In this paper, we have considered the class of operators \( M - A(n^*) \). We have presented some properties of these operators. We also proved that a-Browder’s theorem and the Fuglede-Putnam theorem hold for it.

5-Acknowledgement
Regarding this paper, the authors wish to extend their gratitude to the anonymous referees for their diligent review, insightful feedback, and constructive suggestions.

6. References