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Some theorems on the class M-A(n*) operators on Hilbert space

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Abstract:

An operator $T_1 \in B(H)$ is referred to as $M - A(n^*)$ operators if $(T_1{}^*|T_1|^{2n}T_1)^{\frac{1}{n+1}} - M|T_1{}^*|^2 \geq 0$ for a positive integer n. The well-known Fuglede–Putnam's theorem states that the operator equation $T_1X = XT_2$ implies $T_1{}^*X = XT_2{}^*$ when T_1 and T_2 are normal operators. This work demonstrates that if X is a Hilbert-Schmidt operator, T_1 *belongs to the class* $M - A(n^*)$ operators and $T_2{}^*$ is an invertible operator belonging to *the class* $M - A(n^*)$ operators such that $T_1X = XT_2$, then $T_1{}^*X = XT_2{}^*$.

Keywords: Hilbert space, Fuglede-Putman, A(n*) operators**,**

1- **Introduction:**

Let H denote a separable complex Hilbert space characterized by the inner product $<\cdot,\cdot>$. The space denoted as $B(H)$ represents the set of all bounded linear operators on H, whereas $I = IH$ represents the identity operator.

An operator $T \in B(H)$ is said to be positive (denoted $T \geq O$) if $\langle T\,x, xi\,\rangle \geq 0$ for all $x \in H$. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\bar{\lambda}$. The closure of a set M will be denoted by $\bar{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. We write $\sigma(T), \sigma_p(T)$ *and* $\sigma_a(T)$ for the spectrum, point spectrum and approximate point spectrum, respectively. Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by $iso\sigma(T)$ and $acc\sigma(T)$, respectively.

For an operator T, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - T\,T^*$ (the self–commutator of T). In the following we will mention some known classes of operators defined in Hilbert space H. An operator T, is said to be normal, if $[T^*, T] = 0$, and T is said to be a hyponormal, if $[T^*, T]$ is nonnegative, equivalently if $|T|^2 \geq |T^*|^2$. An operator T is a paranormal if $||T^2x|| \geq ||T\,x||^2$, [4], and it is M–paranormal if $M\,||T^2x|| \geq ||T\,x||^2$ (see [3]) for every unit vector $x \in H$. In [5] authors, Furuta, Ito and Yamazaki introduced the A class of operators, respectively class $A(n)$ of operators defined as follows: for each $n > 0$, an operator T is a class $A(n)$ operator if

$$(T^*|T|^{2n}T)^{\frac{1}{n+1}} \geq |T|^2,$$

(for $n = 1$ it defines the class A operators) which includes the class of log-hyponormal operators (see Theorem 2, in [5]) and it is included in the class of paranormal operators, in case where $n = 1$ (see Theorem 1 in [5]). In the same paper the absolute-n-paranormal operators were introduced as follows: For each n > 0, an operator T is an absolute-n-paranormal operator if

$$\||T|^{n}T\,x\| \geq \|T\,x\|^{n+1} ,$$

for every $n > 0$. In case where $n = 1$ it defines the class A^* operators. Every class A^* operator is a $*$-paranormal operator, Theorem 1.3 in [4]. In paper [10] the absolute-n $*$ -paranormal class of operators was introduced as follows:

$\||T|^{n}T\,x\| \geq \|T^{*}x\|^{n+1}$.

For each $n > 0$, every class $A(n^{*})$ operator is an absolute-n $*$ - paranormal operator, Theorem 2.4 in [10].

Definition 1-1

For each $n > 0, M > 0$ an operator T is a class $M - A(n^{*})$ operator if $(T^{*}|T|^{2n}T)^{\frac{1}{n+1}} \geq M\,|T^{*}|^{2}$.

Any an absolute-n $* - M$-paranormal operator, if for each $n > 0, M > 0$ $\||T|^{n}T\,x\| \geq M\|T^{*}x\|^{n+1}$, for every unit vector $x \in H$ and every class $A(n^{*})$ is an absolute-n $* - M$-paranormal operator [10].

The primary objective of this work is to demonstrate that a-Browder's and Fuglede-Putnam theorems are applicable to $M - A(n^{*})\ operator$.

2. Browder's theorem

If there is a vector $x \neq 0$ that satisfies $(T - \lambda)x = 0$, then a complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator T. If $T \in B(H)$, we may refer to the null space as $N(T)$ and the range as $R(T)$.

The spectrum and the approximate point spectra of T are indicated as $\sigma(T)$ and $\sigma_a(T)$ respectively. The Fredholm operator T is defined as follows: $R(T)$ is closed, $\alpha(T) = dim\,N(T) < \infty$, and $\beta(T) = dim\,H/R(T) < \infty$.

Furthermore, if the $ind(T) = \alpha(T) - \beta(T) = 0$, then T is known as the Weyl operator. The essential spectrum, denoted as $\sigma_e(T)$, and the Weyl spectrum, denoted as $\sigma_w(T)$, are theoretically defined as follows:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \ is\ not\ Fredholm\},$$

and

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \ is\ not\ Weyl\}.$$

An operator $T \in B(H)$ has the finite ascent if $N(T^{m}) = N(T^{m+1})$ for a positive integer m, and finite descent if $R(T^{n}) = R(T^{n+1})$ for a positive integer n. If the operator T is Fredholm of finite ascent and descent, it is referred to be Browder. The Browder spectrum of T may be expressed as:

$$\sigma_b(T) = \{\lambda \in \mathbb{C}: T - \lambda \ \ is\ not\ a\ Browder\ compound\}.$$

The Browder's theorem applies to T if
$\sigma_w(T) = \sigma_b(T)$.

The operator T has the single valued extension property, which is known as SVEP. This property is defined as follows: if $f(z)$ is an analytic vector valued function on an open set $D \subset \mathbb{C}$, such that $(T - \lambda)f(z) = 0$ for all $z \in D$, then $f(z) = 0$ for all $z \in D$.

This section demonstrates that a-Browder's theorem applies to the class $M - A(n^{*})$ operators.

Theorem 2.2

Let $T \in B(H)$ be an operator in the class $M - A(n^{*})$. If $(T - \lambda)x = 0$, then $(T^{*} - \overline{\lambda})\,x = 0$ for all $\lambda \in \mathbb{C}$.

Proof:

Since $\langle M|T^{*}|^{2}x, x\rangle \leq \langle (T^{*}|T|^{2n}T)^{\frac{1}{n+1}}x, x\rangle$=$\langle T^{*}T\,x, x\rangle = |\lambda|^{2}\|x\|^{2}$.

Thus, $\left\|T^{*}x - \bar{\lambda}x\right\|^{2} = \langle T^{*}x - \bar{\lambda}x, T^{*}x - \bar{\lambda}x\rangle$

$=\langle T^{*}x, T^{*}x\rangle - \overline{\lambda}\langle x, T^{*}x\rangle - \overline{\lambda}\langle T^{*}x, x\rangle + |\lambda|^{2}$

$=\langle |T^{*}|^{2}x, x\rangle - \overline{\lambda}\langle Tx, x\rangle - \overline{\lambda}\langle x, Tx\rangle + |\lambda|^{2}$

$$\leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0.$$

Hence, $T^* x = \bar{\lambda} x$.

Lemma 2.3

If T belongs to the class $M - A(n^*)$, then $T - \lambda$ has finite ascent for each $\lambda \in \mathbb{C}$.

Proof:

Since T is a class $M - A(n^*)$ operator, it follows that $N(T - \lambda) \subset N(T^* - \lambda)$, for each $\lambda \in C$ by Theorem 2.2. Therefore, it is possible to express $T - \lambda$ as the following $2x2$ operator matrix in relation to the decomposition $N(T - \lambda) \oplus N(T - \lambda)^{\perp}$:

$$T - \lambda = \begin{bmatrix} 0 & 0 \\ 0 & T_1 \end{bmatrix}.$$

Let $x \in N((T - \lambda)^2)$, and let's write $x = a + b$, where $a \in N(T - \lambda)$ and $b \in N(T - \lambda)^{\perp}$. Then $0 = (T - \lambda)^2 x = (T - \lambda)^2 b$, so that $(T - \lambda) b \in N(T - \lambda) \cap N(T - \lambda)^{\perp} = \{0\}$, which implies that $b \in N(T - \lambda)$, and hence $x \in N(T - \lambda)$. Therefore $N(T - \lambda) = N(T - \lambda)^2$.

Corollary 2.4

If $T \in$ the class $M - A(n^*)$ operator, then T possess the property of SVEP.

Proof:

The Proof may be derived straight from Lemma 2.3 and Proposition 1.8 as presented in reference [8].

In this demonstration, we shall establish the validity of a-Browder's theorem for the class M-A(n*) operator. To do this, we require the subsequent definitions.

Definition 2.5

The Browder essential approximate point spectrum $\sigma_{ab}(T)$ of T is defined by

$$\sigma_{ba}(T) = \cap \{\sigma_a(T + K) : TK = KT, K \text{ is a compact operator}\}.$$

Definition 2.6

We say that a-Browder's theorem holds for T if $\sigma_{ea}(T) = \sigma_{ba}(T)$. It is well known that a-Browder's theorem $\Rightarrow$ Browder's theorem.

Theorem 2.7

Let T ∈ B(H) be a class $M - A(n^*)$ operator. Then T obeys a-Browder's theorem.

Proof:

The SVEP of any operator in the class M-A(n*) operator implies that T satisfies a-Browder's theorem, as stated in Theorem 2.8 in reference [11].

3- Fuglede-Putnam theorem

The use of the Fuglede-Putnam theorem has significant importance within the realm of products, encompassing sums, which are composed of normal operators. An instance illustrating the use of this theory is the Kaplansky theorem [6]. Many scholars in the field of mathematics strive to further develop this theorem within the framework of nonnormal operators (see to [11]).

This is the well-known Fuglede-Putnam theorem, as stated in reference [3].

Theorem 3-1

Consider two operators T_1 and T_2 be normal operator and X be an operator such that $T_1 X = X T_2$, then $T_1{}^* X = X T_2{}^*$.

Assume T is an operator in the space $B(H)$ and $\{e_i\}$ be an orthonormal basis for $H. We\ define$ the Hilbert-Schmidt norm as $\|T\|_2 = (\sum_{i=1}^{\infty}\|Te_i\|^2)^{\frac{1}{2}}$. This definition is independent of the choice of basis (see [2]). If $\|T\|_2 < \infty$, then T is said to be a Hilbert-Schmidt operator and we denote the Hilbert-Schmidt class by $C_2(H)$. The set $C_2(H)$ form an ideal of the algebra $B(H)$. The ideal is a Hilbert space with an inner product $\langle X_1, X_2\rangle = \sum_{i=1}^{\infty}\langle X_1e_i, X_2e_i\rangle = tr({X_2}^*X_1) = tr(X_1{X_2}^*)$. For each pair of operators $T_1, T_2 \in B(H)$, there is an operator Γ_{T_1,T_2} defined on $C_2(H)$ via the formula $\Gamma_{T_1,T_2}(X) = T_1XT_2$ in [3]. Obviously $\|\Gamma\| \leq \|T_1\|\ \|T_2\|$. The adjoint of Γ is obtained by the formula $\Gamma *_{T_1,T_2} X = {T_1}^*X{T_2}^*$, as stated in the reference [1].

Theorem 2.1. If T is an invertible $M - A(n^*)$ operator for $M > 0$, then T^{-1} is also be an absolute $M - A(n^*)$ operator.

Proof:

Given that $(T^*|T|^{2n}T)^{\frac{1}{n+1}} = \left(T^{*(n+1)}T^{(n+1)}\right)^{\frac{1}{n+1}}$, it follows that $(TT^*)^{(n+1)} \leq (T^*T)^{(n+1)}$.

$$(T^*T)^{\frac{-(n+1)}{2}}((T^*T)^{(n+1)} - (TT^*)^{(n+1)})(T^*T)^{\frac{-(n+1)}{2}} \geq 0.$$

Thus, $(T^*T)^{\frac{-(n+1)}{2}}(TT^*)^{(n+1)}(T^*T)^{\frac{-(n+1)}{2}} \leq I$ and $(T^*T)^{\frac{(n+1)}{2}}(TT^*)^{-(n+1)}(T^*T)^{\frac{(n+1)}{2}} \geq I$.

$$(T^*T)^{\frac{(n+1)}{2}}((TT^*)^{-(n+1)} - (T^*T)^{-(n+1)})(T^*T)^{\frac{(n+1)}{2}} \geq 0.$$

This is equivalent to $(TT^*)^{-(n+1)} - (T^*T)^{-(n+1)} = \left({T^{-1}}^*T^{-1}\right)^{(n+1)} - \left(T^{-1}{T^{-1}}^*\right)^{(n+1)} \geq 0.$

The class $M - A(n^*)$ operator includes the operator T^{-1}.

Theorem 3-2

If T_1, T_2 and ${T_2}^*$ are $M - A(n^*)$ operators, then the operator Γ_{T_1,T_2} belongs to $M - A(n^*)$ operators' class.

Proof:

Since $\Gamma *_{T_1,T_2} \Gamma_{T_1,T_2}X = {T_1}^*T_1XT_2{T_2}^*$ and $\Gamma_{T_1,T_2}\Gamma *_{T_1,T_2} X = T_1{T_1}^*X{T_2}^*T_2$ for any operator X in $C_2(H)$.

We get $|\Gamma_{T_1,T_2}|X = |T_1|X|{T_2}^*|$ and $|\Gamma *_{T_1,T_2}|X = |{T_1}^*|X|T_2|$

As well as, $|\Gamma_{T_1,T_2}|^2X = |T_1|^2X|{T_2}^*|^2$and $|\Gamma *_{T_1,T_2}|^2X = |{T_1}^*|^2X|T_2|^2$.

We have $|\Gamma_{T_1,T_2}|^{2n}X = |T_1|^{2n}X|{T_2}^*|^{2n}$ and $|\Gamma *_{T_1,T_2}|^{2n}X = |{T_1}^*|^{2n}X|T_2|^{2n}$ for each $n > 0$.

Thus, $\left(\Gamma *_{T_1,T_2} |\Gamma_{T_1,T_2}|^{2n}\Gamma_{T_1,T_2}\right)X = ({T_1}^*|T_1|^{2n}T_1)X({T_2}^*|{T_2}^*|^{2n}T_2)$

$$\geq (|{T_1}^*|^2)^{n+1}X\,(|T_2|^2)^{n+1}$$
$$= \left(|\Gamma *_{T_1,T_2}|^2\right)^{n+1}X.$$

Theorem 3-3

Let $T_1$and T_2 be $M - A(n^*)$ operator such that ${T_2}^*$ is invertible operator in the class of $M - A(n^*)$ operators, and let X be a Hilbert-Schmidt operator. If $T_1X = XT_2$, then ${T_1}^*X = X{T_2}^*$.

Proof:

Let Γ_{T_1,T_2} be the Hilbert-Schmidt operator defined by $\Gamma_{T_1,T_2}X = T_1X(T_2)^{-1}$.

Since T_1, T_2 are in the class $M - A(n^*)$ operators, by Theorem 3-2, $\Gamma_{T_1,T_2}$is $M - A(n^*)$ operator. The hypothesis $T_1X = XT_2$, implies that $T_1X({T_2}^*)^{-1} = X$.

And Γ_{T_1,T_2} X = X , $\Gamma *_{T_1,T_2} X = X$ by Theorem.

Hence, we have ${T_1}^*X\left({T_2}^{-1}\right)^* = X$. Therefore, ${T_1}^*X = X{T_2}^*$.

4-Conclusion

In this paper, we have considered the class of operators $M - A(n^*)$. We have presented some properties of these operators. We also proved that a-Browder's theorem and the Fuglede-Putnam theorem hold for it.

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