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A Strong form of Cl- Cl-Connectedness

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Abstract: S. Modak and T. Noiri introduced a weak form of connectedness[1] in topological space called Cl- Cl-connectedness. In this paper, we introduce the notion of θsCl- θsCl-separated sets and θsCl- θsCl-connectedness spaces which is strong form of Cl- Cl-connectedness. We obtain several properties of this notion.

Keywords: Cl- Cl**-separated,** Cl- Cl**-connected,** θ**-semi-open,** θ **−semi-closed,** θsCl-θsCl**-separated,** θsCl-θsCl**-connected.**

1. Introduction

In 2016, S. Modak and T. Noiri introduced a weak form of connectedness[1] in topological space. In this paper, we introduce a further strong form of Cl- Cl-connectedness, this form is said to be θsCl-θsCl-connectedness. We investigate several properties of θsCl-θsCl-connectedness spaces analogous to connected spaces and also, we show that every connected space is θsCl- θsCl-connected. We show that θsCl-θsCl-connectedness is preserved under continuous functions. Let (X,τ) or X be a topological space or space and $A \subseteq X$. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$. A subset A of X is said to be semi-open[2] if $A \subseteq Cl(Int(A))$. A subset B of X is semi-closed if its compliment is semi-open. Intersection of all semi-closed sets containing A is called the semi-closure[3,4] of A and it is denoted by $sCl(A)$. The collection of all semi-open subsets of X denoted by $SO(X,\tau)$. We set $SO(X,x) = \{U: x \in U \in SO(X,\tau)\}$.

2. θsCl-θsCl-SEPARATED SETS

DEFINITION.2.1. [5] A point $x \in X$ is said to be a θ-semi-cluster point of a subset S of X if $Cl(U) \cap A \neq \phi$ for every $U \in SO(X,x)$. The set of all θ-semi-cluster points of A is called the θ-semi-closure of A and denoted by $\theta sCl(A)$. A subset A is called θ-semi-closed if $A = \theta sCl(A)$. The compliment of a θ-semi-closed set is called θ-semi-open.

LEMMA.2.2. [5] For the θ-semi-closure of a subset A of a topological space (X,τ), the following properties are hold:

1.$A \subset sCl(A) \subset \theta sCl(A)$ and $sCl(A) = \theta sCl(A)$ if $A \in SO(X)$.

2.$A \subset B, then\ \theta sCl(A) \subset \theta sCl(B)$.

3.$\theta sCl(\theta sCl(A)) = \theta sCl(A)$ and $\theta sCl(A)$ is θ-semi-closed set.

DEFINITION.2.3. [1] Let be a topological space, nonempty subsets A , B of X are called Cl-Cl-separated sets if $Cl(A) \cap Cl(B) = \phi$.

DEFINITION.2.4. Let be a topological space, nonempty subsets A , B of X are called θsCl-θsCl-separated sets if $\theta sCl(A) \cap \theta sCl(B) = \phi$.

From above definitions, we have following implications:

$$Cl\text{-}Cl\text{- separated} \Rightarrow \theta sCl\text{-}\theta sCl\text{-separated}$$

But the converse need not hold in general:-

EXAMPLE.2.5. In $\mathbb{R}$ be the real line with the usual topology on $\mathbb{R}$ the sets $A = (0,1)$ and $B = (1,2)$ are θsCl-θsCl-separated but not Cl-Cl- separated.

THEOREM.2.6. Let A and B be $\theta sCl - \theta sCl$-separated in a space X, if $C \subset A$ and $D \subset B$, then C and D are also θsCl-θsCl-separated.

Proof: Obvious.

3. θsCl-θsCl-CONNECTED SETS

DEFINITION.3.1. [1] A subset A of a space X is said to be Cl- Cl-connected if A is not the union of two Cl- Cl-separated sets in X.

DEFINITION.3.2. A subset A of a space X is said to be θsCl-θsCl-connected if A is not the union of two θsCl-θsCl-separated sets in X.

From above definitions we have following implications:

$$\theta sCl\text{-}\theta sCl\text{-connected} \Rightarrow Cl\text{-}Cl\text{- connected}$$

However the converse need not hold in general:-

EXAMPLE.3.3. Let $\mathbb{R}$ be the real line with the usual topology on $\mathbb{R}$. Let $X = (3,4) \cup (4,5)$. Consider the sets $A = (3,4) \cap X$ and $B = (4,5) \cap X$ are θsCl-θsCl-separated but not Cl-Cl-separated, since $X = A \cup B$. Hence X is Cl-Cl- connected but not θsCl-θsCl-connected.

DEFINITION.3.4. A subset A of X is s-V-θ-connected if it cannot be expressed as the union of nonempty subsets with disjoint θ −semi-closed neighbourhood in X, that is if there are no disjoint nonempty sets C and D and no open sets U and V such that $A = C \cup D$, $C \subset U, D \subset V$ and $\theta sCl(U) \cap \theta sCl(V) = \phi$.

THEOREM.3.5. Every s-V-θ-connected space is a θsCl-θsCl-connected space.

Proof: The proof is obvious from Definition 3.3.

Following examples show that T_0-space and θsCl-θsCl-connected space are independent concept.

EXAMPLE.3.6. Let $X = \{a, b\}$, $\tau = \{\phi, \{a\}, \{b\}, X\}$. Then $X = \{a\} \cup \{b\}$ and $\theta sCl\{a\} \cap \theta sCl\{b\} = \phi$. The space is a T_0-space but X is not a θsCl-θsCl-connected space.

EXAMPLE.3.7. Let X be a set containing more than one points, and τ be the indiscrete topology on X. Then (X, τ) is not a T_0-space but it is a θsCl-θsCl-connected space.

THEOREM.3.8. A space X is θsCl-θsCl-connected if and only if it cannot be expressed as the disjoint union of two nonempty clopen sets.

Proof: Let X be a θsCl- θsCl-connected space, if possible suppose that $X = W_1 \cup W_2$, where $W_1 \cap W_2 = \phi$, W_1 *and* W_2 are nonempty clopen sets in X. Since W_1 *and* W_2 are clopen sets in X, then $\theta sCl(W_1) \cap \theta sCl(W_2) = \phi$. Therefore X is not a θsCl-θsCl-connected space. This is a contradiction.

Conversely, suppose that $X \neq W_1 \cup W_2$ and $W_1 \cap W_2 = \phi$, where W_1 *and* W_2 are nonempty clopen sets in X. We shall prove that X is a θsCl-θsCl-connected space, if possible suppose that X is not a X is a θsCl-θsCl-connected space then there exist θsCl-θsCl-separated sets A and B such that $X = A \cup B$. Then $X = \theta sCl(A) \cup \theta sCl(B)$ and $\theta sCl(A) \cap \theta sCl(B) = \phi$, set $W_1 = \theta sCl(A)$ and $W_2 = \theta sCl(B)$. Then W_1 *and* W_2 are nonempty clopen sets. Moreover, we have $X = W_1 \cup W_2$ and $W_1 \cap W_2 = \phi$. This is a contradiction, so X is a θsCl-θsCl-connected space.

THEOREM.3.9. Let X be a space, if A is a θsCl-θsCl-connected subset of X and H, G are θsCl-θsCl-separated subsets of X with $A \subset H \cup G$ then either $A \subset H$ *or* $A \subset G$.

Proof: Let A be a θsCl-θsCl-connected set. Let $A \subset H \cup G$, since $A = (A \cap H) \cup (A \cap G)$, then $\theta sCl(A \cap H) \cup \theta sCl(A \cap G) \subset \theta sCl(H) \cap \theta sCl(G) = \phi$. Suppose $A \cap H$ *and* $A \cap G$ are nonempty. Then A is not θsCl-θsCl-connected. This is a contradiction. Thus, either $A \cap H = \phi$ *or* $A \cap G = \phi$. This implies that $A \subset H$ *or* $A \subset G$.

THEOREM.3.10. If A and B are θsCl-θsCl-connected sets of a space X and if A and B are not θsCl- θsCl-separated, then $A \cup B$ is a θsCl- θsCl-connected.

Proof: Let A and B be θsCl-θsCl-connected sets in X. Suppose $A \cup B$ is not θsCl-θsCl-connected. Then, there exist two nonempty disjoint θsCl-θsCl-separated sets G *and* H such that $A \cup B = G \cup H$. Suppose that $\theta sCl(G) \cap \theta sCl(H) = \phi$. Since A *and* B are θsCl- θsCl-connected, by Theorem.3.8, either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$.

Case(i) If $A \subset G$ and $B \subset H$, then $A \cap H = B \cap G = \phi$. Therefore, $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = (A \cap G) \cup \phi = A \cap G = A$. Also, $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = (B \cap H) \cup \phi = B \cap H = B$. Now, $\theta sCl(A) \cap \theta sCl(B) = \theta sCl$ $((A \cup B) \cap G) \cap \theta sCl$ $((A \cup B) \cap H) \subset \theta sCl(H) \cap \theta sCl(G) = \phi$. Thus, A *and* B are θsCl-θsCl-separated, which is a contradiction. Hence, $A \cup B$ is θsCl- θsCl-connected.

Case(ii) If $B \subset G$ and $A \subset H$, then $B \cap H = A \cap G = \phi$. Therefore, $(A \cup B) \cap H = (A \cap H) \cup (B \cap H) = (A \cap H) \cup \phi = A \cap H = A$. Also, $(A \cup B) \cap G = (A \cap G) \cup (B \cap G) = (B \cap G) \cup \phi = B \cap G = B$. Now, $\theta sCl(A) \cap \theta sCl(B) = \theta sCl$ $((A \cup B) \cap H) \cap \theta sCl$ $((A \cup B) \cap G) \subset \theta sCl(H) \cap \theta sCl(G) = \phi$. Thus, A *and* B are θsCl-θsCl-separated, which is a contradiction. Hence, $A \cup B$ is θsCl- θsCl-connected.

THEOREM.3.11. If $\{M_i : i \in I\}$ is a nonempty family of θsCl-θsCl-connected sets of a space X, with $\cap_{i \in I} M_i \neq \phi$, then $\cup_{i \in I} M_i$ is θsCl-θsCl-connected.

Proof: Suppose $\cup_{i \in I} M_i$ is not θsCl-θsCl-connected. Then we have $\cup_{i \in I} M_i = G \cup H$, G *and* H are θsCl- θsCl-separated sets in X. Since $\cap_{i \in I} M_i \neq \phi$, we have a point $x \in \cap_{i \in I} M_i$. Since $x \in \cup_{i \in I} M_i$, either $x \in G$ *or* $x \in H$. Suppose that $x \in G$, since $x \in M_i$ for each $i \in I$, then M_i *and* G intersect for each $i \in I$. By Theorem.3.8, $M_i \subset G$ *or* $M_i \subset H$. Since G *and* H are disjoint, $M_i \subset G$ for all $i \in I$ and hence $\cup_{i \in I} M_i \subset G$. This implies that H *is empty*. This is a contradiction. Suppose that $x \in H$. By Similar way, we have that G *is empty*. This is a contradiction. Thus, $\cup_{i \in I} M_i$ is θsCl-θsCl-connected.

THEOREM.3.12. Let X be a space, $\{A_\alpha : \alpha \in \Delta\}$ be a family of θsCl-θsCl-connected sets and A be a θsCl-θsCl-connected set, if $A \cap A_\alpha \neq \phi$ for every $\alpha \in \Delta$, then $A \cup (\cup_{\alpha \in \Delta} A_\alpha)$ is θsCl-$vsCl$-connected.

Proof: Since $A \cap A_\alpha \neq \phi$ for each $\alpha \in \Delta$, by Theorem.3.10, $A \cup A_\alpha$ is θsCl-θsCl-connected for each $\alpha \in \Delta$. Moreover, $A \cup (\cup A_\alpha) = \cup (A \cup A_\alpha)$ and $\phi \neq A \subset \cap (A \cup A_\alpha)$. Thus by Theorem.3.10, $A \cup (\cup_{\alpha \in \Delta} A_\alpha)$ is θsCl-θsCl-connected.

4. θsCl-θsCl-CONNECTEDNESS AND MAPPINGS

THEOREM.4.1. The continuous image of a θsCl-θsCl-connected space is a θsCl-θsCl-connected space.

Proof: Let $f : X \longrightarrow Y$ be a continuous map and X be a θsCl-θsCl-connected space. If possible suppose that $f(X)$ is not a θsCl-θsCl-connected subset of Y. Then, there exist nonempty θsCl-θsCl-separated sets A and B such that $f(X) = A \cup B$. Since f is continuous and $\theta sCl(A) \cap \theta sCl(B) = \phi$, $\theta sCl(f^{-1}(A)) \cap \theta sCl(f^{-1}(B)) \subset$ $f^{-1}((\theta sCl(A)) \cap f^{-1}(\theta sCl(B)) = f^{-1}(\theta sCl(A) \cap \theta sCl(B)) = \phi$. Since A *and* B are nonempty, thus $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ are θsCl- θsCl-separated and $X = f^{-1}(A) \cup f^{-1}(B)$. This contradicts that X is θsCl- θsCl-connected. Therefore, $f(X)$ is a θsCl- θsCl-connected.

THEOREM.4.2. Let X be a space then following are equivalent conditions:

1.X is not θsCl-θsCl-connected.

2.$X = W_1 \cup W_2$, where $W_1 \cap W_2 = \phi$, W_1 *and* W_2 are nonempty clopen sets in X.

3.There is a continuous map $f : X \longrightarrow (Y, \sigma)$ such that $f(x) = 0$ if $x \in W_1$ and $f(x) = 1$ if $x \in W_2$, where $Y = \{0,1\}$ and σ is the discrete topology on Y.

Proof: (1) $\Leftrightarrow$ (2) Obvious from Theorem.3.7.

(2)$\Rightarrow$ (3) Let $Y = \{0,1\}$ and σ is the discrete topology on Y, then (Y, σ) is the topological space. Let $f: X \longrightarrow (Y, \sigma)$ be a function defined by $f(W_1) = 0$ and $f(W_2) = 1$. Then f is a continuous surjection such that $f(x) = 0$ for each $x \in W_1$ and $f(x) = 1$ for each $x \in W_2$.

(3)$\Rightarrow$ (2) Here $W_1 = f^{-1}(0)$ is a clopen set of X and $W_2 = f^{-1}(1)$ is a clopen set of X, and also X is a disjoint union of nonempty sets W_1 and W_2.

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