A Generalized Andre Plane of order 3^4

K.Annapurna[1], Sarada Kesiraju[2]

Former Head, Department of Mathematics

Head, Department of Mathematics and Statistics

RBVRR Women's College, Narayanguda, Hyderabad -500027,T.S.

e-mail[1] : annapurnakamaraju@yahoo.com

e-mail[2]: sarada.koride@gmail.com

Summary: A translation plane of order 3^4 is constructed. It is shown that the plane is a generalized Andre plane and computed the translation complement of the plane .It is found to be of order 6400.

1. Introduction

Rao and Davis have given [14] the construction of translation planes through t-spread sets. It may be recalled that Bruck and Bose have contributed to the theory of t-spread sets over finite fields for the construction of non Desarguesian translation planes through their papers [1] , [2] . In this paper we have constructed a translation plane π of order 3^4 using 3- spread sets . This plane is shown to be a generalized Andre plane applying the technique given by D.A.Foulser [6]. By making use of the properties of the collineation groups of the plane π the translation complement of the plane π is computed and found to be of order 6400.

2. Description of the plane π and identifying the plane as a generalized Andre system

It is well known that a translation plane π of finite order can be coordinatized by a V-W system. Conversely given a V-W system (Q,+, ·) a translation plane π(Q) can be associated with Q [8,pp 362]. A V-W system can be constructed from a t-spread set.[1,pp95]. Thus the construction of translation plane of order q^{t+1} reduces to the construction of t-spread set. [3,,pp220]

t-spread set : Let t be a positive integer . A set $\mathscr{C}$ of (t+1) by (t+1) matrices over F is a t-spread set over F if it satisfies

a) $|\mathscr{C}| = q^{t+1}$, $\mathscr{C}$ contains the zero and identity matrices.
b) For all X, Y ∈ $\mathscr{C}$, X ≠ Y => det (X-Y) ≠ 0.

Here det A denotes the determinant of the matrix A.

Through out this paper F, (abcd,efgh klmn,pqrs) and i.p denote the Galois Field GF(3) ,the 4x4 matrix $\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ p & q & r & s \end{pmatrix}$ and ideal point respectively.

For M,N ∈ GL(4,3), T(M, N) = {A∈GL(4,3) | $A^{-1}MA=N$} , .Z(M) = T(M,M).

Let G denote the translation complement of the translation plane π; G_0 (G_{81}) denotes the collinetion subgroup of G fixing the i.p 0 (81); $G_{0,81}$ denotes the (autotopism) collineation subgroup of G fixing the i.ps

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0,81 and $G_{0,81,1}$ denotes the subgroup of G (conjugation collineation group) fixing the i.ps 0,81,1;In general $G_{i,j,k,l,m}$ denotes the collineation subgroup of G fixing the i.ps i, j,k ,l ,m.

The translation plane π under study is constructed through a 3- spread set $\mathscr{C}$ over F .The spread set $\mathscr{C}$ is given by

$$\mathscr{C} = \{\mathbf{0}\} \cup A_0 \mathscr{G} \cup A_1 \mathscr{G} \cup A_2 \mathscr{G} \cup A_3 \mathscr{G}$$

where $\mathscr{G} = < X,Y / X,Y \in GL(4,3) , X^5 = I, Y^2 = -I ,Y^{-1}XY = X^{-1} >$ is a meta cyclic group of order 20 in GL(4,3) where X = (2120,0212,2221,1022) , Y = (1100,1200,2012,1222)

and
A_0 = (1000,0100,0010,0001)
A_1 = (0001,0011,0121,1001)
A_2 = (0010,0001,1100,0110)
A_3 = (0100,1100,1101,1112)

Table 1

I	M_i	C.P of M_i		I	M_i	C.P of M_i
0	(0000,0000,0000,0000)			41	(0010,0001,1100,0110)	[0121]
1	(1000,0100,0010,0001)	[2021]		42	(1112,2011,1001,1200)	[2211]
2	(1210,0121,1112,2011)	[2121]		43	(0102,2210,0221,1122)	[0121]
3	(2201,1020,0102,2210)	[1111]		44	(1110,0111,1111,1211)	[2101]
4	(1010,0101,1110,0111)	[2121]		45	(2110,0211,1121,1212)	[1101]
5	(2022,2102,2110,0211)	[1111]		46	(0020,0002,2200,0220)	[0111]
6	(2000,0200,0020,0002)	[1011]		47	(2221,1022,2002,2100)	[1221]
7	(2120,0212,2221,1022)	[1111]		48	(0201,1120,0112,2211)	[0111]
8	(1102,2010,0201,1120)	[2121]		49	(2220,0222,2222,2122)	[1101]
9	(2020,0202,2220,0222)	[1111]		50	(1220,0122,2212,2121)	[2101]
10	(1011,1201,1220,0122)	[2121]		51	(2012,1222,2000,0212)	[0001]
11	(1100,1200,2012,1222)	[0201]		52	(1221,1002,2120,2010)	[0001]
12	(1001,1122,1221,1002)	[0201]		53	(2021,2202,1102,0202)	[0001]
13	(0221,1211,2021,2202)	[0201]		54	(0022,0021,2020,1201)	[0001]
14	(1111,1212,0022,0021)	[0201]		55	(0210,1202,1011,0100)	[0001]
15	(1121,0220,0210,1202)	[0201]		56	(1021,2111,1000,0121)	[0001]
16	(2200,2100,1021,2111)	[0201]		57	(2112,2001,1210,1020)	[0001]
17	(2002,2211,2112,2001)	[0201]		58	(1012,1101,2201,0101)	[0001]
18	(0112,2122,1012,1101)	[0201]		59	(0011,0012,1010,2102)	[0001]
19	(2222,2121,0011,0012)	[0201]		60	(0120,2101,2022,0200)	[0001]
20	(2212,0110,0120,2101)	[0201]		61	(0100,1100,1101,1112)	[0001]
21	(0001,0011,0121,1001)	[0001]		62	(0121,1001,0012,0102)	[0001]
22	(2011,0120,1020,0221)	[0001]		63	(1020,0221,2101,1110)	[0001]
23	(2210,2012,0101,1111)	[0001]		64	(0101,1111,1222,2110)	[0001]
24	(0111,1221,2102,1121)	[0001]		65	(2102,1121,1002,0020)	[0001]
25	(0211,2021,0200,2200)	[0001]		66	(0200,2200,2202,2221)	[0001]
26	(0002,0022,0212,2002)	[0001]		67	(0212,2002,0021,0201)	[0001]
27	(1022,0210,2010,0112)	[0001]		68	(2010,0112,1202,2220)	[0001]
28	(1120,1021,0202,2222)	[0001]		69	(0202,2222,2111,1220)	[0001]
29	(0222,2112,1201,2212)	[0001]		70	(1201,2212,2001,0010)	[0001]

30	(0122,1012,0100,1100)	[0001]		71	(1200,2000,0222,0120)	[0001]
31	(1222,0201,0110,2022)	[0001]		72	(1122,2120,0122,2012)	[0001]
32	(1002,2220,1200,2000)	[0001]		73	(1211,1102,0001,1221)	[0001]
33	(2202,1220,1122,2120)	[0001]		74	(1212,2020,2011,2021)	[0001]
34	(0021,0010,1211,1102)	[0001]		75	(0220,1011,2210,0022)	[0001]
35	(1202,1112,1212,2020)	[0001]		76	(2100,1000,0111,0210)	[0001]
36	(2111,0102,0220,1011)	[0001]		77	(2211,1210,0211,1021)	[0001]
37	(2001,1110,2100,1000)	[0001]		78	(2122,2201,0002,2112)	[0001]
38	(1101,2110,2211,1210)	[0001]		79	(2121,1010,1022,1012)	[0001]
39	(0012,0020,2122,2201)	[0001]		80	(0110,2022,1120,0011)	[0001]
40	(2101,2221,2121,1010)	[0001]		81	-----	

The translation plane π under study is constructed through the 3-spread set $\mathscr{C}$ over F by considering

4-dimensional subspaces V_i , $0 \le i \le 81$ of V(8,3) , the 8-dimensional vector space over F as follows:

Let $V_i = \{(x,y) \ / \ y = x M_i , x \in F^4 \}$, $0 \le i \le 80$, $V_{81} = \{ (0,y) / y \in F^4\}$.The incidence structure whose points are vectors of $V = F^8$ and whose lines are V_i , $0 \le i \le 81$ and their cosets in the additive group of V with inclusion as incidence relation is the translation plane π associated with the 3-spread set $\mathscr{C}$.

Left and Middle nuclei of the t-spread set : If $\mathscr{C}$ is a t-spread set then

$$M_\lambda = \{ M \in \mathscr{C} \mid \mathscr{C} M = \mathscr{C} \}$$

$$M_\mu = \{ M \in \mathscr{C} \mid M\mathscr{C} = \mathscr{C} \}$$

Left nucleus M_λ and middle nucleus M_μ are multiplicative groups of GL(t+1,q) and if $M \in \mathscr{C}$ and $M^2 \notin \mathscr{C}$

Then $M \notin M_\lambda \cup M_\mu$. It can be observed that the left and middle nuclei of the above 3-spread set $\mathscr{C}$ are as given below

Left Nucleus : $M_\lambda = \mathscr{G} \cup A_1\mathscr{G} = < M_3 , M_{21} \mid M_3^5 = I, M_{21}^8 = I, M_{21}^{-1} M_3 M_{21} = M_3^3 >$

Left nucleus is a meta cyclic group of order 40.

Middle nucleus M_μ: $M_\mu = \mathscr{G} \cup A_3\mathscr{G} = < M_3 , M_{61} \mid M_3^5 = I, M_{61}^8 = I, M_{61}^{-1} M_3 M_{61} = M_3^3 >$

Middle nucleus is also a meta cyclic group of order 40.

V-W system associated with the spread set $\mathscr{C}$

Let $(Q,+,\cdot)$ be a system constructed from the 3-spread set $\mathscr{C}$ where $Q = F^4$, the operation ' +' is the ordinary vector sum. Let e = (1000). For each $y \in Q$ there is a unique matrix $M \in \mathscr{C}$ (denoted by M(y)) such that

y = e M. For $x, y \in Q$, $y \neq 0$ define y.x= x M(y) and 0.x=0. The system $(Q,+,\cdot)$ is a left V-W system coordinatizing the translation plane π. Let N_λ, N_μ be the left and middle nuclei of the V-W system $(Q,+,\cdot)$

So $N_\lambda = <$ (1210),(0001) $>$ and $N_\mu = <$ (1210),(0100) $>$. $N_\lambda \cap N_\mu$ contains a unique cyclic subgroup generated by g of order 10 where g= (1210).

V-W system is a λ -system:

The quadruples of Q are indexed as follows $Q = \{x_i \mid x_i = e M_i , M_i \in \mathscr{C} , 0 \le i \le 80\}$.$x_2$=(1210)=g

We observe the following:

$$X_{10j+1} \,.\, g = g^{3^{\lambda(X_{10j+1})}} \,.\, X_{10j+1} \Leftrightarrow g\, M(X_{10j+1}) = X_{10j+1}\, M\left(g^{3^{\lambda(X_{10j+1})}}\right)$$

$$\Leftrightarrow g\, A_{j_1}\, M_{11}^{k} M_2^{i-1} = e\, A_{j_1}\, M_{11}^{k} M_2^{i-1} M_2^{3^{\lambda(X_{10j+1})}}$$

$$\Leftrightarrow g\, A_{j_1}\, M_{11}^{k} = e\, A_{j_1}\, M_{11}^{k}\, M_2^{3^{\lambda(X_{10j+1})}}$$

$$\Leftrightarrow g = e\,(A_{j_1}\,(M_{11}^{k}\, M_2^{3^{\lambda(X_{10j+1})}}\, M_{11}^{-k})\, A_{j_1}^{-1})$$

$$\Leftrightarrow \lambda(X_{10j+1}) = 0 \quad \text{where } j=0,4$$
$$= 2 \quad j=1,5$$
$$= 1 \quad j=2,6 \qquad \text{I}$$
$$= 3 \quad j=3,7$$

For $0 \le j \le 7, 1 \le i \le 10$ define $\lambda(X_{10j+i}) = \lambda(X_{10j+1})$ II

Now we see $X_{10j+i} \,.\, g = g^{3^{\lambda(X_{10j+i})}} \,.\, X_{10j+i} \Leftrightarrow gM(X_{10j+i}) = X_{10j+i}\, M\left(g^{3^{\lambda(X_{10j+1})}}\right)$

$$\Leftrightarrow g\, A_{j_1}\, M_{11}^{k} M_2^{i-1} = e\, A_{j_1}\, M_{11}^{k} M_2^{i-1} M_2^{3^{\lambda(X_{10j+1})}}$$

$$\Leftrightarrow g\, A_{j_1}\, M_{11}^{k} = e\, A_{j_1}\, M_{11}^{k}\, M_2^{3^{\lambda(X_{10j+1})}}$$

$$\Leftrightarrow X_{10j+1} . g = g^{3^{\lambda(X_{10j+1})}} \,.\, X_{10j+1}$$

From the above it is clear that the mapping $\lambda : Q^* \rightarrow Z_4$ (integers modulo 4) defined in I and II satisfy the property $x.\, g = g^{3^{\lambda(x)}} .x$ for all $x \in Q^*$. [By theorem in 13 pp 541] V-W system is a λ- system.

3. Collineations of the translation plane π

Any non- singular linear transformation on $V=F^8$ induces a collineation of π fixing the point corresponding to the zero vector if and only if the linear transformation permutes the subspaces V_i , $0 \le i \le 81$ among themselves. Equivalently , a non singular linear transformation $T=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$, where B,C,D and E are 4x4 matrices over F, induces a collineation of π fixing the point corresponding to the zero vector if and only if the following conditions (a) and (b) are satisfied.[17,Theorem 1]

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(a) If D is non-singular ,then $D^{-1}E \in \mathscr{C}$. if D is singular then D is the zero matrix and E is non-singular.

(b) For M $\in \mathscr{C}$ if (B+MD) is non-singular ,then $(B+MD)^{-1}(C+ME) \in \mathscr{C}$ if (B+MD) is singular then (B+MD) is the zero matrix and (C+ME) is non- singular.

The group of all collineations leaving the point corresponding to the zero vector of π invariant is called the translation complement of π. Through out this paper, by a collineation we mean a collineation from the translation complement of π.

3.1 Collineations corresponding to the Left and middle nuclei

The mappings $\alpha = \begin{bmatrix} I & 0 \\ 0 & M_2 \end{bmatrix}, \beta = \begin{bmatrix} I & 0 \\ 0 & M_{11} \end{bmatrix}, \lambda = \begin{bmatrix} I & 0 \\ 0 & M_{21} \end{bmatrix}, \gamma_1 = \begin{bmatrix} M_2^{-1} & 0 \\ 0 & I \end{bmatrix}$

$\gamma_2 = \begin{bmatrix} M_{61}^{-1} & 0 \\ 0 & I \end{bmatrix}$ are all collineations of π and the actions of the collineations α , β on the set of i.ps. of π are furnished below:

α : (0)(81)(1,2,…,10)(11, 12,…,20)(21,22,…,30)(31,32,…,40)

(41 ,42,…,50)(51,52,…,60)(61,62,…,70)(71,72,…,80)

β : (0)(81)(1,11,6,16)(2,20,7,15)(3,19,8,14)(4,18,9,13)(5,17,10,12)

(21,31,26,36)(22,40,27,35)(23,39,28,34)(24,38,29,33)(25,37,30,32)

(41,51,46,56)(42,60,47,55)((43,59,48,54)(44,58,49,53)(45,57,50,52)

(61,71,66,76)(62,80,67,75)(63,79,68,74)(64,78,69,73)(65,77,70,72)

Also $\lambda^{-1}\alpha\lambda = \alpha^3$, $\lambda^{-1}\beta\lambda = \beta\alpha^8$, $\gamma_i^{-1}\alpha\,\gamma_i = \alpha$, $\gamma_i^{-1}\beta\,\gamma_i = \beta$, i = 1,2.

The actions of the collineations $\lambda, \gamma_1, \gamma_2$ on the set of i.ps of π are computed and furnished below.

λ : (0)(81)(1,21,12,32,6,26,17,37)(2,24,11,39,7,29,16,34) (3,27,20,36,8,22,15,31)

(4,30,19,33,9,25,14,38) (5,23,18,40,10,28,13,35)(22,60,40,47,27,55,35,42)

(23,51,31,48,28,56,36,43)(41,62,52,71,46,67,57,76) (42,65,51,78,47,70,56,73)

(43,68,60,75,48,63,55,80) (44,61,59,72,49,66,54,77)(45,64,58,79,50,69,53,74)

γ_1 : (0)(81)(1,2,3,4,5,6,7,8,9,10)(11,20,19,18,17,16,15,14,13,12)

(21,24,27,30,23,26,29,22,25,28) (31,38,35,32,39,36,33,40,37,34)

(41,42,43,44,45,46,47,48,49,50)(51,60,59,58,57,56,55,54,53,52)

(61,64,67,70,63,66,69,62,65,68)(71,78,75,72,79,76,73,80,77,74)

γ_2 : (0)(81)(1,61,11,71,6,66,16,76)(2,62,12,72,7,67,17,77) (3,63,13,73,8,68,18,78)

(4,64,14,74,9,69,19,79) (5,65,15,75,10,70,20,80)(21,59,39,46,26,54,34,41)

(25,53,33,50, 30,58,38,45)

Let $\gamma_3 = \gamma_2{}^2$. Now $\gamma_3 = \begin{bmatrix} M_{11}^{-1} & 0 \\ 0 & I \end{bmatrix}$ is a collineation of π and its action on the set of i.ps of π follows from the action of γ_2 and is given below:

γ_3 : (0)(81) (1,11,6,16)(2,12,7,17)(3,13,8,18)(4,14,9,20)(5,15,10,20) (21,32,26,37) (22,33,27,38) (23,34,28,39) (24,35,29,40) (25,36,30,31)(41,59,46,54) (42,60,47,55) (43,51,48,56) (44,52,49,57) (45,53,50,58) (61,80,66,75) (62,71,67,76) (63,72,68,77) (64,73,69,78) (65,74,70,79)

Homology groups; [9,pp 385]: From the left nucleus of the plane and the collineations α, λ it is clear that < α , λ > is the ((∞),[0,0])- homology group H_1 of π. From the middle nucleus and the collineations γ_1, γ_2 of π

$< \gamma_1 , \gamma_2 >$ is the ((0), [0])-homology group H_2 of π.Both homology groups are meta cyclic groups of order 40.The collineation group $< H_1 , H_2 > = < \alpha,\lambda, \gamma_1, \gamma_2 >$ divides the set of i.ps of π into three orbits $\mathcal{O}_i$, i = 1,2,3 of lengths 1, 1, 80 where $\mathcal{O}_1 = \{0\}$, $\mathcal{O}_2 = \{81\}$, $\mathcal{O}_3 = \{ i \mid 1 \leq i \leq 80 \}$.

3.2 Conjugacy collineations of the plane

A mapping $\delta = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$, where $A \in GL(4,3)$ induces a conjugation collineation of π if $A^{-1} \mathscr{C} A = \mathscr{C}$. The set of all conjugation collineations of π forms a group called the conjugation collineation group, and this group fixes the ideal points corresponding to V(0),V(∞), and V(I). Conjugacy collineations of the plane keeps the left and middle nuclei of $\mathscr{C}$ invariant. From Table 1 the matrices M_i ,i = 3,5,7,9 are the only matrices with C.P [1111] and the matrices M_i , i = 41,43 are the only matrices with C.P [0121]. So every collineation either fixes the i.p 41 or flips the i.ps 41,43 while keeping the set of i.ps S={ 3,5,7.9 } invariant. In order to keep the set of i.ps of S invariant under δ the matrix A of δ belong to the following sets:

$K_1 = Z(M_3) \cap Z(M_{41})$ $\qquad$ $K_4 = Z(M_3) \cap T(M_{41},M_{43})$

$K_2= T(M_3 , M_9) \cap Z(M_{41})$ $\qquad$ $K_5 = T(M_3 , M_9) \cap T(M_{41},M_{43})$

$K_3 = T (M_3,M_7) \cap Z(M_{41})$ $\qquad$ $K_6 = T(M_3 , M_7) \cap T(M_{41},M_{43})$

The sets K_2, K_3, K_4, K_6 are empty. $K_1 = Z(M_3)$, $K_5 = T(M_3,M_9)$. No conjugacy collineation maps the i.p 3 onto the i.p 7 and every conjugation collineation either fixes the i.ps 3 and 41 or flips the i.ps 3, 9 and 41, 43.
Also since $M_\lambda \cap M_\mu = \mathscr{G}$. each matrix of $\mathscr{G}$ induces a conjugacy collineation .Let $\delta_1 = \gamma_1{}^{-1}\alpha$, $\delta_2 = \gamma_3{}^{-1}\beta$. The mappings δ_1 and δ_2 are collineations of π since they are product of collineations. $< \delta_1, \delta_2 >$ is a subgroup of $G_{0,81,1}$ and is isomorphic to $\mathscr{G}$
The actions of the conjugation collineations δ_1 , δ_2 on the set of i.ps. of π can be computed from the actions of α , $\beta, \gamma_1, \gamma_3$ and are furnished below:

δ_1: (0)(81)(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11,13,15,17,19)(12,14,16,18,20) (21,29,27,25,23) (22,30,28,26,24) (31,35,39,33,37)(32,36,40,34,38)(41)(42)(43)(44) (45)(46)(47)(48)(49)(50) (51,53,55,57,59)(52,54,56,58,60) (61,69,67,65,63) (62,70,68,66,64) (71,75,79,73,77)

(72,76,80,74,78)

δ_2: (0)(81)(1)(2,10)(3,9)(4,8)(5,7)(6)(11)(12,20)(13,19)(14,18)(15,17)(16)
(21,23)(22)(24,30)(25,29)(26,28)(27)(31,39)(32,38)(33,37)(34,36)(35)(40)
(41,43)(42)(44,50)(45,49)(46,48)(47)(51,59)(52,58)(53,57)(54,56)(55)(60)
(61)(62,70)(63,69)(64,68)(65,67)(66)(71)(72,80)(73,79)(74,78)(75,77)(76)

From the actions of the collineations δ_1 and δ_2 on the set of i.ps. of π it is clear that the collineation group $<\delta_1, \delta_2>$ is transitive on the set of i.ps. {3,9},{11,13,15,17,19}and {12,14,16,18,20}separately.

If $\delta_3 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ is a mapping fixing the i.ps. 3 and 11 then the matrix A of δ_3 belongs $Z(M_3) \cap Z(M_{11})$, where

$$Z(M_3) \cap Z(M_{11}) = \left\{ A(a,d) = \begin{pmatrix} a & 2d & 2d & d \\ d & a+d & 2d & 2d \\ 2d & 0 & a+d & 2d \\ 2d & d & 0 & a+d \end{pmatrix} \mid (a,d) \neq (0,0), a, d \in F \right\}$$

If $A=A(1,1)^2$ = (0112,2211,1021,1202) then $A^{-1}M_{21}A=M_{26}$, $A^{-1}M_{41}A=M_{41}$, $A^{-1}M_{61}A=M_{66}$

If A of δ is such that $A^{-1}\,\mathcal{G}A = \mathcal{G}$ then δ induces a collineation of π fixing the i.ps 0,81,1if and only if $A^{-1}M_{20i+1}A$, $1 \leq i \leq 3$ belong to distinct left cosets of $\mathcal{G}$ .Hence $A=A(1,1)^2$ induces a conjugation collineation.

Also all even powers of A(1,1) induce conjugation collineations and the odd powers of A(1,1) do not induce conjugation collineations. So $G_{0,81,1,3,11} = <\delta_3>$ where $\delta_3= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $A=A(1,1)^2$. δ_3 is of order 4.

The action of the collineation δ_3 on the set of i.ps. of π is computed and furnished below:

δ_3: (0)(81)(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12)(13)(14)(15)(16)(17)(18)(19)(20)(21,26)
(22,27)(23,28)(24,29)(25,30)(31,36)(32,37)(33,38)(34,39)(35,40)(41)(42)(43)(44)(45)(46)
(47)(48)(49)(50)(51)(52)(53)(54)(55)(56)(57)(58)(59)(60)(61,66)(62,67)(63,68)(64,69)(71,76)
(73,78)(74,79)(75,80).

If $\delta_4 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is a mapping fixing the i.p. 3 and mapping the i.p. 11 onto the i.p.12 then the

matrix A of δ_4 belongs to

$$Z(M_3) \cap T(M_{11},M_{12}) = \left\{ A(c,d) = \begin{pmatrix} 2c+2d & 2c & c & d \\ d & 2c & 2c & c \\ c & c+d & 2c & 2c \\ 2c & 0 & c+d & 2c \end{pmatrix} \mid c, d \in F, (c,d) \neq (0,0) \right\}$$

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It may be seen that no matrix of this set keeps $\mathscr{C}$ invariant under conjugation. From this no matrix of this set yields a conjugation collineation and thus no conjugation collineation of π which fixes the i.p 3 maps the i.p 11onto the i.p 12.

$$G_{0,81,1,3} = G_{0,81,1,3,11} \cup \{ \bigcup_{i=1}^{5} G_{0,81,1,3,11}\, \delta_1^{\,i} \} = < \delta_1 , \delta_3 >$$

$$\left| G_{0,81,1,3} \right| = 5 \left| G_{0,81,1,3,11} \right| = 5x4=20$$

If $\delta_5 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is a mapping flipping the i.ps 3,9 and mapping the i.p 11 onto the i.p 12 then the matrix A of δ_5

belongs to $T(M_3,M_9) \cap T(M_{11},M_{12})$. Analysing as in the earlier case no matrix of this set yields a conjugation collineation. From this we conclude that no conjugation collineation which flips the i.ps 3 and 9 maps the i.p 11 onto the i.p 12.Hence the collineation group $G_{0,81,1}$ is transitive on the set of i.ps {3,9} and {11,13,15,17,19} separately. Now

$$G_{0,81,1} = G_{0,81,1,3} \cup G_{0,81,1,3}\, \delta_2 = < \delta_1 , \delta_2 , \delta_3 >$$

Since $G_{0,81,1}$ is transitive on {3,9}

$$\left| G_{0,81,1,} \right| = 2 \left| G_{0,81,1,3} \right| = 2x20=40$$

In the above discussion we have seen that the collineation group $G_{0,81,1,3,11}$ flips the i.ps 21 and 26. If

$\delta_6 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is a mapping that fixes the i.ps 0,81,1,3,11 and 21 then the matrix A of δ_6 belongs to

$Z(M_3) \cap Z(M_{11}) \cap Z(M_{21})$.By a straight forward computation we see that $A= \pm I$. The mapping δ_6 induces a scalar collineation which fixes all the i.ps .and

$$G_{0,81,1,3,11,21} = < \delta_6 > \cong F^*$$

Thus $G_{0,81,1,3,11,21}$ is the ((0,0),[∞])- homology group H_3 of π and hence gives the kernel of π.Thus the kernel K of Q is isomorphic to F .Hence the kernel of π is trivial.

The collineation group $< \alpha, \lambda, \gamma_1, \gamma_2 >$ fixes the i.ps 0 and 81 and is transitive on $\mathscr{O}_3$.

$$G_{0,81} = \left\{ \bigcup_{i=1}^{20} G_{0,81,1}\xi_i \right\} \cup \left\{ \bigcup_{i=1}^{20} G_{0,81,1}\lambda\xi_i \right\} \cup \left\{ \bigcup_{i=1}^{20} G_{0,81,1}\lambda\gamma_2^{-1}\xi_i \right\} \cup \left\{ \bigcup_{i=1}^{20} G_{0,81,1}\gamma_2\xi_i \right\}$$ where ξ_i is a collineation from the collineation group $< \alpha,\beta >$ mapping the i.p 1 onto the i.p i ,$1 \le i \le 20$ while fixing the i.ps 0 and 81.

$$G_{0,81} = < \delta_1, \delta_2, \alpha, \beta, \lambda, \gamma_2 > = < \alpha, \lambda, \gamma_2 \,||, \gamma_1 > \text{ since } \beta = \lambda^2\alpha^{-1} \text{ and } < \delta_1,\delta_2 > \subset < \alpha, \lambda, \gamma_1, \gamma_2 >$$

$| G_{0,81} | = 80 \; | G_{0,81,1} | = 80x40 = 3200$

3.3 Translation complement of π

Let $\theta = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ where A = (1000,0001,0011,0121). It may be seen that $\theta : M \to A^{-1}M^{-1}A$, M ∈ $\mathscr{C}$ and

$V_0\,\theta = V_{81}$, $V_{81}\,\theta = V_{0.}$ Further

$\theta : M_2 \to M_2^7 , M_{11} \to M_{17}$, $A_1 = M_{21} \to M_{29}$, $A_2 = M_{41} \to M_{50}$, $A_3 = M_{61} \to M_{37}$

For $0 \le j \le 7, 1 \le i \le 10$ we have

$\theta : M_{10j+i} \to A^{-1}M_2^{\,1-i}M_{11}^{\,-k}\, A_{j_1}^{-1}\, A$ where $j_1 = \left[\frac{j}{2}\right]$, $k = j - 2j_1$

$= (A^{-1} M_2^{\,1-i} A)(A^{-1} M_{11}^{\,-k} A)(A^{-1} A_{j_1}^{-1} A)$

$= M_2^{7(i-1)} M_{17}^{k} (A^{-1} A_{j_1}^{-1} A)$

For various values of j; j_1 takes the values 0,1,2,3. When j = 0 or 1 then $j_1 = 0$ and

$\theta : M_{10j+i} \to M_2^{7(i-1)}M_{17}^k \in \mathscr{C}$. When j = 2 or 3 then $j_1 = 1$ and $\theta : M_{10j+i} \to M_2^{7(i-1)}M_{17}^k M_{79} \in \mathscr{C}$

since $< M_2 , M_{11}> \subset M_{\mu}$. When j = 4 or 5 , $\theta : M_{10j+i} \to M_2^{7(i-1)}M_{17}^k M_{50} \in \mathscr{C}$ since $\mathscr{G} \subset M_{\mu}$.

Also when j = 6 or 7, θ sends M_{10j+i} onto $M_2^{7(i-1)}M_{17}^k M_{37} \in \mathscr{C}$ as M_2 , $M_{11} \in M_{\mu}$. This shows that θ permutes the non zero matrices of $\mathscr{C}$ among themselves. From this it follows that θ is a collineation of π flipping the i.ps 0,81 and fixing the i,p. 1.

It may be seen that $V_0\,\theta = V_{81}$, $V_{81}\,\theta = V_0$, $V_1\,\theta = V_1$, $V_{21}\,\theta = V_{79}$, $V_{41}\,\theta = V_{50}$,$V_{61}\,\theta = V_{37}$ and $\theta^{-1}\alpha\,\theta = \gamma_1^7$, $\theta^{-1}\beta\,\theta = \gamma_2^{-1}\,\gamma_1^{-1}$. From these relations we get the following:

$V_{1+i}\,\theta = V_{1+k_1}$ where $k_1 \equiv 7i \;(mod 10)$

$V_{11+i}\,\theta = V_{11+k_2}$ where $k_2 \equiv 3i + 6 \;(mod 10)$

$V_{21+i}\,\theta = V_{71+k_3}$ where $k_3 \equiv 9i + 8 \;(mod 10)$

$V_{31+i}\,\theta = V_{61+k_4}$ where $k_4 \equiv i + 5 \;(mod 10)$

$V_{41+i}\,\theta = V_{41+k_5}$ where $k_5 \equiv 7i + 9 \;(mod 10)$

$V_{51+i}\,\theta = V_{51+k_6}$ where $k_6 \equiv 3i + 3 \;(mod 10)$

$V_{61+i}\,\theta = V_{31+k_7}$ where $k_7 \equiv 9i + 6 \;(mod 10)$

$V_{71+i}\,\theta = V_{21+k_4}$

The action of the collineation θ on the set of i.ps. of π is now given by

θ : (0,81)(1)(2,8,10,4)(3,5,9,7)(11,17,15,19)(12,20,14,16)(13)(18)(21,79,24,76) (22,78,23,77)

(25,75,30,80)(26,74,29,71)(27,73,28,72)(31,66,32,67)(35,68,40,65)(34,69,39,64)

(35,70,38,63)(36,61,37,62)(41,50,43,44)(42,47)(45,48,49,46) (51,54,53,60)(52,57)(55,56,59,58)

It may be noted that the collineation group < $G_{0,81}$, θ > divides the set of i.ps. into two orbits $\mathscr{O}_1$ = {0,81} and

$\mathscr{O}_2$ = { i | 1≤ i ≤ 80 }. Further $G_{0,81}\,\theta$ gives the set of all collineations of π that flips the i.ps. 0 and 81.

3.4 Non existence of certain collineations

Lemma 1. (a) No collineation of π maps the i.p. 1 onto the i.p. 81(0) and the i.p.81(0) onto the i.p. 0(81).

(b) No collineation of π maps the i.p. k onto the i.p. 81(0) and the i.p.81(0) onto the i.p. 0(81).

(c) Every collineation that fixes the i.p. 0(81) also fixes the i.p.81(0) i.e.,$G_0 = G_{81} = G_{0,81}$.

Proof. If π has a collineation which maps the i.p.1 onto the i.p. 81 and the i.p.81 onto the i.p. 0 then by a Lemma [16,chapter 3] $M+M_1 \in \mathscr{C}$ for all $M \in \mathscr{C}$. This condition does not hold since

$M_{21} + M_1$ = (1001,0111,0101,1002) $\notin \mathscr{C}$. This shows that no collineation of π maps the i.p. 1 onto the i.p. 81 and the i.p.81 onto the i.p. 0.

If μ is a collineation with the following action

$$\mu: (1,0.81,\ldots,\ldots,\ldots)$$

Then $\theta^{-1}\,\mu\,\theta$ maps the i.p. 1 onto the i.p.81 and the i.p. 81 onto the i.p. 0 – a contradiction to the above. This proves the first part of the lemma.

Let μ_1 (μ_2) be a collineation mapping the i.p. k onto the i.p.81(0) and the i.p. 81(0) onto the i.p.0(81). Since $G_{0,81}$ fixes the i.ps. 0 and 81 and is transitive on the remaining i.ps., there exists a collineation $\tau \in G_{0,81}$ which maps the i.p. 1 onto the i.p. k. Then $\tau\mu_1\,\tau^{-1}(\tau\mu_2\,\tau^{-1})$ maps the i.p. 1 onto the i.p. 81(0) and the i.p. 81(0) onto the i.p. 0(81) – a contradiction to the first part of the lemma. We have already seen that π is a λ – plane and Q is a λ – system with proper kern. By a result of Foulser [6.pp.390] No collineation of π fixes the i.p. 0 (81) and moves the i.p.81(0) i.e., every collineation of π that fixes the i.p.0 also fixes the i.p.81.

Therefore $G_0 = G_{81} = G_{0,81}$. This proves the last part of the lemma.

Lemma 2 Every Collineation of π either fixes both the i.ps. 0 and 81 or flips them.

Proof. In view of the above lemma no collineation of π maps the i.p. k onto the i.p. 0(81) via the i.p.81(0), k≠ 0 (k ≠ 81), every collineation of π that fixes the i.p.81 also fixes the i.p.0 and vice versa.

Assume that ξ is a collineation of π mapping the i.ps. 0 and 81 onto any two i.ps. other than 0 and 81. Using the transitivity of $G_{0,81}$ on the set of i.ps. other than the i.ps.0 and 81 we can take without loss of generality that $\boldsymbol{\xi}$ maps the i.p. 0 onto the i.p. 1. This plane π has a collineation θ which fixes the i.p. 1 and moves the remaining i.ps. except 6,13,18.

Now in view of lemma [16,chapter 3] no collineation of π moves the i.p. 0 onto the i.p. 1 and the i.p.81 onto the i.p. k, where k $\notin$ {6,13,18}. But π has a collineation δ_2 which fixes the i.p. 1 moves both the i.ps. 13 and 18 and again by the same lemma [16,chapter 3] we get that no collineation of π maps the i.p.0 onto the i.p. 1 and the i.p.81 onto either the i.p. 13 or the i.p.18.Thus the collineation ξ must map the i.p.81 onto the i.p. 6 while mapping the i.p.0 onto the i.p. 1. By a result of Maduram [12,pp 487]. the spread sets $\mathscr{C}$ and $\mathscr{C}_{1,6,k}$ must be conjugate where k ≠ 1,6 and

$$\mathscr{C}_{1,6,k} = [\ N_i = \{(M_i - M_6)^{-1} - (M_1 - M_6)^{-1}\} - \{(M_k - M_6)^{-1} - (M_1 - M_6)^{-1}\} \mid M_i \in \mathscr{C}]$$

i.e., $\mathscr{C}_{1,6,k}$ is a 3-spread set of π with V_1, V_6 and V_k as the fundamental subspaces (**y = 0, x = 0, y = x**).It may be observed that every matrix of $\mathscr{C}$ is of det 1 and the spread set $\mathscr{C}_{1,6}$ contains matrices of determinants 1 and 2. The matrices N_0, $N_1 \in \mathscr{C}_{1,6,0}$ are of det 1 and N_4 is of det 2 (table .2). It now follows that the spread sets $\mathscr{C}_{1,6,k}$ for any k ≠ 1,6 contains matrices of det 1 and det 2. Thus the spread sets $\mathscr{C}$ and $\mathscr{C}_{1,6,k}$ are not conjugate for any k – a contradiction. From this it follows that no collineation of π maps the i.p. 0 onto the i.p. 1 and the i.p.81 onto the i.p. 6. Thus no collineation of π moves both the i.ps. 0 and 81 outside the orbit {0, 81}.Therefore every collineation of π either fixes both the i.ps. 0 and 81 or flips them.

Table 2

i	$M_i - M_6$	$(M_i - M_6)^{-1}$	$X_i = (M_i - M_6)^{-1} - (M_1 - M_6)^{-1}$	$N_i = X_i X_i^{-1}$	Det N_i
0	(1000,0100,0010,0001)	(1000,0100,0010,0001)	(2000,0200,0020,0002)	(1000,0100,0010,0001)	1
1	(2000,0200,0020,0002)	(2000,0200,0020,0002)	0	0	0
2	(2210,0221,1122,2012)	(2211,1021,1202,2020)	(0211,1121,1212,2021)	(0122,2212,2121,1012)	1
3	(0201,1120,0112,2211)	(2012,2101,1010,0101)	(0012,2201,1020,0102)	(0021,1102,2010,0201)	2
4	(2010,0201,1120,0112)	(2101,1010,0101,1110)	(0101,1110,0111,1111)	(0202,2220,0222,2222)	2
5	(0022,2202,2120,0212	(2110,0211,1121,1212)	(0110,0011,1101,1210)	(0220,0022,2202,2120)	1
6	(0000,0000,0000,0000)	∞	∞	∞	--

Theorem: The translation complement G of the translation plane π is given by G = $< \alpha, \lambda, \gamma_1, \gamma_2, \theta >$. It is of order 6400 and divides the set of i.ps. into two orbits of lengths 2 and 80 where the small orbit consists of the i.ps. 0 and 81.

Proof: The collineation group $< \alpha, \lambda, \gamma_1, \gamma_2, \theta >$ is a subgroup of G. If ξ is any collineation of π then by the above lemma 2, ξ either belongs to $G_{0,81}$ or $G_{0,81}\,\theta$.

Thus $\xi \in < G_{0,81}, \theta > = < \alpha, \lambda, \gamma_1, \gamma_2, \theta >$. Thus G = $< \alpha, \lambda, \gamma_1, \gamma_2, \theta >$ and G divides the set of i.ps. of π into two orbits of lengths 2 and 80, where the smaller orbit consists of the i.ps.0 and 81.Since G flips 0 and 81

$$| G | = 2 | G_0 | = 2 | G_{0,81} | = 2x\ 3200 = 6400.$$

Hence the theorem.

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