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β_0-Connectedness in Topological Spaces

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Abstract: *In this paper, we introduce a new form of connectedness in topological space called β_0-connectedness which is stronger form of β-connectedness and weaker form of connectedness. We also investigate some special properties of β_0-connectedness. We further consider the components of β_0-connectedness.*

Keywords: β-open, β_0-open, connectedness, β-connectedness, β_0-connectedness, topological spaces.

Introduction

Several form of connectedness were investigated in the literature such as semi-connectedness [7] preconnectedness [5] and β-connectedness [1,3,6] based on the notions, semi-open [8], preopen [4], semipreopen [9] (β-open [2]) sets respectively. In this paper we introduce a new type of connectedness called β_0-connectedness which is stronger form of β-connectedness and weaker form of connectedness.

This paper organized as follows. Section-2 develops the necessary preliminaries. The concept of β_0-open sets is introduced. Section-3 introduces the notion of β_0-connected spaces. It is shown that β_0-connectedness lie between β-connectedness and connectedness. Section-4 contains the properties of β_0-connected sets. Section-5 studies the concept of β_0-components.

Preliminaries

In this section we will discuss some relevant definitions and introduce some new results for the paper. Let (X, τ) be a topological space, we will denote Cl(A) and Int(A) the closure of A and the interior of A respectively. We also denote X is a topological space in place of (X, τ).

Definition 2.1: A subset A of X is said to be open (resp. β-open [1]. if $A \subseteq IntA$ (resp. $A \subseteq Cl(Int(Cl(A)))$.
The complement of open (resp. β-open) set is said to be closed (resp. β-closed).

Definition 2.2: A β-open subset of A of a topological space X is said to be β_0-open if for each x ∈ A there exist a closed set F such that $x \in F \subseteq A$. A subset B of a topological space X is β_0-closed, if X\B is β_0-open.

The family of all open (resp. β-open, β_0-open) subsets of a topological space X is denoted by $O(X)(resp.\ \beta_O(X),\ \beta_O O(X))$. The family of all closed (resp. β-closed, β_0-closed) subsets of a topological space X is denoted by C(X) (resp., βC(X), β_0C(X)).

Definition 2.3: A point $x \in X$ is said to be an β_0-interior point of A, if there exists an β_0-open set U containing x such that $x \in U \subseteq A$. The set of all β_0-interior points of A is said to be β_0-interior of A and it is denoted by $\beta_0$int(A).

Definition 2.4: Intersection of all β_0-closed sets containing F is called β_0-closure of F and it is denoted by β_0Cl(A).

It may be noted that β_0-open sets are obtained from β-open sets but the collection of these sets is neither a sub-collection of open sets nor it contains the collection of open sets. Thus, the study of β_0-open set is meaningful.

Example2.5: Let $X = \{a, bc, d\}$, $\tau = \{\phi,\ X,\ \{a\}, \{b\}, \{a,b\}, \{b,c,d\}\}$. Here $\{a,c,d\} \in \beta_0 O(X)$ but $\{a,c,d\} \notin \tau$. Also $\{b\} \in \tau$ but $\{b\} \notin \beta_0 O(X)$.

Lemma 2.6: Let A be a subset of a topological space X. If A is both open and closed, thus A is both β_0-open and β_0-closed.

Proof: Let A be a subset of X w2hich is both open and closed in X. Then $int(A) = A = Cl(A)$ and $A \subseteq Cl(\mathrm{int}(Cl(A)))$. Thus A is β-open. Let B = X \A is also β-open and A and B are closed set. Thus A and B are β_0-open set which implies A is β_0-opne and β_0-closed.

The converse of the above lemma need not hold in general:

Example 2.7: In $\mathbb{R}$ with the usual topology on $\mathbb{R}$ the set (*a*, *b*) is β_0-open and β_0-called, but not open and closed in $\mathbb{R}$.

Definition 2.8: Let X and Y be two topological spaces. A function $f : X \to Y$ is β_0-contains at a point $x \in X$ if for each open set V of Y containing v, there exists an β_0-open set U in X containing x such that $f(U) \subseteq V$. If f is β_0-continuous at every point x of X, then it is called β_0-continuous.

Proposition 2.9: Let X and Y be two topological space. A function $f : X \to Y$ is β_0-continuous if and only if the inverse image of every open set in Y is β_0-open in X.

Proof: Sufficiency: Let V be an open set of Y. Let x be a point of $f^{-1}(V)$. Then $f(n) \in V$, so by hypothesis there is β_0-open set U_x containing 'x' such that $f(U_x) \subset V$. Then $U_x \subset f^{-1}(V)$.

Thus $f^{-1}(V)=\bigcup_{x\in f^{-1}(v)} U_x$ is β_0-open in X.

Necessity: Let $x \in X$ and V be an open set in Y containing $f(x)$. Then $f^{-1}(V)$ is β_0-open and $x \in f^{-1}(V)$. Since $f(f^{-1}(v)) \subseteq V$, f is β_0-continuous.

3 β_0-Connected Space

Definition 3.1: Nonempty subsets A and B of a topological space X are said to β_0-separated if $A\cap\beta_0 Cl(B)=\phi=\beta_0 Cl(A)\cap B$

It is obvious that two β_0-separated sets are disjoint. If A and B are two β_0-separated sets in X with $\phi\neq C\subset A$ and $\phi\neq D\subset B$, then C and D are also β_0-separated sets in *X*.

Following is the existence of β_0-separated sets:

Example 3.2: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {a, b, c}, {a, b, d}, {a, c, d}, X}. We take A = {a, c, d} and B = {b}.

Then $\beta_0 Cl(A) = A$ and $\beta_0 Cl(B) = B$. Therefore $A\cap\beta_0 Cl(B)=\phi=\beta_0 Cl(A)\cap B = A\cap B=\phi$. Thus A and B are two β_0-separated subsets of X.

Definition 3.3: A subset S of a topological space X is said to be β_0-connected if S is not the union of two β_0-separated sets in X.

Existence of β_0-connected sets:

Example 3.4: Let X = {a, b, c, d}, τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X} then $\beta_0 O(X)=\{\phi, \{c,d\}, \{b,c,d\}, X\}$. We cannot express X s the union of two β_0-separated sets in X and so X is β_0-connected.

Theorem 3.5: For a topological space X, the following statements hold :-

(1) If X is β-connected, then X is β_0-connected.

(2) If X is β_0-connected, then X is connected.

Proof: (1) The proof is obvious from the fact that $\beta_0 O(X)\subseteq\beta O(X)$.

(2) Let X be β_0-connected. Then only subsets of X which are both β_0-open and β_0-closed in X are s and X. If possible suppose that S is not connected. Then there erxists a non-empty proper subset A of X which is both open and closed ion X. Then by Lemma 2.6, A is also β_0-open and β_0-closed in X. Thus A is nonempty proper of X which is both β_0-open and β_0-closed in X a contradiction. Hence we must have X is connected.

We give a moderate figure for relationship of various connectedness:

β-connected
↓
β_0-connected
↓
Semi-connected → connected ← preconnected

The converse implications of the above figure need not hold in general.

Example 3.6:
Let $X = \{a,b,c,d\}$ *and* $\tau = \{\phi,\{a\},\{b\},\{a,b\},\{a,b,c\},X\}$. *Then* $\beta 0(X) = \{\phi,\{a\},\{b\},\{a,b\},\{a,c\},\{a,b\},\{,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{b,c$
$\{\phi,\{c,d\},\{b,c,d\},X\}$.

Then X is β_0-connected but not β-connected. Since {a, b} and {b, c} are β-separated sets of X.

Results 3.7: If possible suppose that every connected space is β_0-connected. As every β_0-connected space is β-connected, then every connected space is β_0-connected, then every connected space is β-connected. This contradicts example [3]. Thus every β_0-connected space is not necessarily a connected space.

For remaining counter examples of above figure see [3].

Theorem 3.8: A topological space X is β_0-connected if and only if X cannot be expressed as the union of two disjoint nonempty β_0-open subsets of X.

Proof: Let X be β_0-connected. Let U and V be two disjoint nonempty β_0-Open subsets of X such that X = U∪U. But A = X-U and B = X-V. Then A and B are β_0-closed in X. Thus $A\cap\beta_0 cl(B)=\phi=\beta_0 cl(A)\cap B$ and $X=A\cup B$ is not β_0-connected. Thus is a contradiction. Thus X cannot be expressed as the union of two disjoint nonempty β_0-open subsets of X.

Conversely suppose that the condition holds. Suppose $X=A\cup B,\ A\neq\phi\neq B$ and $A\cap\beta_0 cl(B)=\phi=\beta_0 cl(A)\cap B$. But $U=X\setminus\beta_0 cl(A)$ and $V=X-\beta_0 cl(B)$. Then U and V are nonempty β_0-open sets and $U\cup V=(X-\beta_0 cl(A))\cup(X-\beta_0 cl(B)) = X-(\beta_0 cl(A)\cap\beta_0 cl(B))\subseteq X$. This implied that $X=U\cup V$. Again $U\cap V=(X-\beta_0 cl(A))\cap(X-\beta_0 cl(B))=X-(\beta_0 cl(A)\cup\beta_0 c$. Since $X=A\cup B$. This is a contradiction. Thus X is β_0-connected.

Theorem 3.9: For a topological space X the following statements following are equivalent:

(1) X is β_0-connected.

(2) The only subsets of X which are both β_0-open and β_0-closed are X and the empty set.

(3) X cannot be expressed as the union of two disjoint nonempty β_0-open sets.

(4) There is no non-constant onto β_0-continuous function from X to a discrete space which contains more than one point.

Proof: (1) → (2)

Let X be β_0-connected. Let $A\subset X$ which is both β_0-open and β_0-closed in X. Then B = X – A is also β_0-open and β_0-closed in X. Since A and B are β_0-closed therefore $\beta_0 cl(A)=A$ and $\beta_0 cl(B)=B$. Therefore $\beta_0 cl(A)\cap B=A\cap B=\phi$. Since X is β_0-connected so we

must have one of A and B be empty on X.
(2) $\rightarrow$ (3) Obvious.
(3) $\rightarrow$ (4) . Let Y be a discrete space with more than one point and let $f: X \rightarrow Y$ be an onto β_0-continuous function. Let $Y = U \cup V$, where U and V are two disjoint nonempty β_0-open sets in Y. Since $f: X \rightarrow Y$ is onto0, therefore $f(X) = Y = U \cup V \Rightarrow X = f^{-1}(Y) = f^{-1}(U) \cup f^{-1}(V)$. Since the topology of Y is discrete, so both U and V are open in Y. Again since f is β_0-continuous so the invese image of every open set in Y is β_0-open in X. Consequently $f^{-1}(U)$ and $f^{-1}(V)$ both are (nonempty) β_0-open in X which contradicts (3).
(4) $\Rightarrow$ (1) if possible suppose that X is not β_0-connected. We decompose X as $A \cup B$, where Az and B are nonempty subsets of X such that $\beta_0 cl(A) \cap B = \phi$ or $\beta_0 cl(B) \cap A = \phi$. We see that both A and B are β_0-open sets in X. In fact $B = X - \beta_0 cl(A)$ and $\beta_0 cl(A)$ is the smallest β_0-cclosed set containing A and hence β_0-closed in X. So B is β_0-open in X. Let Y = {0, 1} with discrete topology. We define a map $f: X \rightarrow Y$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \in B. \end{cases}$$

Then $f^{-1}(\phi) = \phi$, which is β_0-open in X, $f^{-1}(Y) = f^{-1}(\{0\} \cup \{1\}) = f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = A \cup B = X$, $f^{-1}(\{0\}) = A$ which is β_0-open in X, $f^{-1}(\{1\}) = B$, which is β_0-open in X. Again $\phi, \{0\}, \{1\}, X$ are open in Y = {0, 1} with discrete topology. Thus we see that the inverse image of every open set in Y is β_0-open in X. Hence f is β_0-continuous and onto which contradicts (4) for X.

Definition 3.10: Let X and Y be two topologies space A function $f: X \rightarrow Y$ is said to be β_0-irresolute if the inverse image of every β_0-open set in Y under f is β_0-open in X.

Theorem 3.11: Let X and Y be two topological spaces. Let $f: X \rightarrow Y$ be an onto β_0-irresolute function. If X is β_0-connected, then Y is β_0-connected.

Proof: Let X and Y be two topological space and $f: X \rightarrow Y$ be an onto β_0-irresolute function. Suppose that X is β_0-connected. If A is a subset of Y which is both β_0-open and β_0-closed, then $f^{-1}(A)$ is both β_0-open and β_0-closed in X. Since X is β_0-separated so $f^{-1}(A)$ must be all of X or the empty set. Therefore $A = f(X) = Y$ or $A = \phi$ and we have proved that Y to be β_0-connected.

Theorem 3.12: Let f : X $\rightarrow$ $\mathbb{R}$ be a β_0-continuous map from a β_0-connected space X to the real line $\mathbb{R}$ if x, y are two points of X such that $a = f(x)$ *and* $b = f(y)$, then every real number 'r' between a *and* b is attained at a point in X.

Proof: Suppose there is no point $c \in X$, such that $f(c) = r$. Then $A = (-\infty, r)$ and $B = (r, \infty)$, are disjoint open sets in $\mathbb{R}$. Since if is β_0-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint β_0 –open sets in X and X= $f^{-1}(A) \cup f^{-1}(B)$. This is a contradiction to the fact that X is β_0-connected. Therefore, there exists a point $c \in X$ such that $f(c) = r$.

4 Properties of β_0-connected sets

Theorem 4.1: If A is a β_0-connected set of a topological space X and U, V are β_0-separated sets of X such that $A \subseteq U \cup V$, then either $A \subseteq U$ or $A \subseteq V$.

Proof: Since $A = (A \cap U) \cup (A \cap V)$, we have $(A \cap U) \cap \beta_0 cl(A \cap V) \subset U \cap \beta_0 cl(V) = \phi$. Similarly we have $(A \cap U) \cap \beta_0 cl(A \cap V) = \phi$. If $A \cap U$ and $A \cap V$ are nonempty, then A is not β_0-connected which is a contradiction. Therefore, either $A \cap U = \phi$ or $A \cap V = \phi$. It follows that either $A \subseteq U$ or $A \subseteq V$.

Theorem 4.2: If A is a β_0-connected set of a topology space X and $A \subseteq N \subseteq \beta_0 cl(A)$, then N is β_0-connected.

Proof: Assume that N is not β_0-connected. Then there exist β_0-separated sets U and V such that $N = U \cup V$. Therefore, U and V are nonempty $U \cap \beta_0 cl(V) = \phi = \beta_0 cl(U) \cap V$. By Theorem 4.1, we obtain either $A \subset U$ or $B \subset V$.

(i) Suppose that $A \subset U$. Then $\beta_0 cl(A) \subset \beta_0 cl(U)$ and $V \cap \beta_0 cl(A) = \phi$. But by hypothesis, $V \subset N \subset \beta_0 cl(A)$ and $V = \beta_0 cl(A) \cap V = \phi$. This is contradiction since V is nonempty.

(ii) Suppose that $A \subset V$. Similarly we obtain that U is empty. This is a contradiction. This implies that N is β_0-connected.

Corollary 4.3: If A is a β_0-connected subset of a topological space X, then $\beta_0 cl(A)$ is β_0-connected.

Theorem 4.3: Let A and B be subsets of a topological space X. *If A and B* are β_0-connected and not β_0-separted in X, then $A \cup B$ is β_0-connected.

Proof: Suppose that $A \cup B$ is not β_0-connected. Then there exist β_0-separated 1, D in X such that $A \cup B = C \cup D$. Then $A \subset C \cup D$. From Theorem 4.1, Other $A \subset C$ or $A \subset D$. Similalry, we obtain that either $B \subset C$ or $B \subset D$. If $A \subset C$ and $B \subset C$, then $A \cup B \subset C$ and D =ϕ. This is a contradiction. Therefore $A \subset C$ and $B \subset D$. Again similarly $A \subset D$ and $B \subset C$. Therefore we obtain $\beta_0 cl(A) \cap B \subset \beta_0 cl(C) \cap n = \phi$ and $\beta_0 cl(B) \cap A \subset \beta_0 cl(C) \cap D = \phi$. Therefore A, B are β_0-separtated in X. This is a contradiction, Therefore, $A \cup B$ is β_0-connected.

Theorem 4.5: If $\{B_r \mid \gamma \in \Gamma\}$ is a nontmpty family of β_0-connected subsets of A topological space X such that $\bigcap_{\gamma \in \Gamma} B_r \neq \phi$, then $\bigcap_{\gamma \in \Gamma} B_r$ is a β_0-connected.

Proof: Suppose that $N = \bigcap_{\gamma\in\Gamma} B_r$ and N is non β_0-connected it follows that $N = U \cup V$, where U and V are β_0-separated sets in X. Since $\bigcap_{\gamma\in\Gamma} B_r \neq \phi$, we can choose a point γ in $\bigcap_{\gamma\in\Gamma} B_r$. Since $n \in N$, either $x \in U$ or $x \in V$.

(1) Suppose that $x \in U$. Since x $\in B_r$ for each $\gamma \in \Gamma$. Now by Theorem 4.1, B_γ must be either U or V. Since U and V are disjoint, $B_\gamma \subset U$ for all $\gamma \in \Gamma$ and hence $N \subset U$. This means that V is empty which is a contradiction.

(2) Suppose x ∈ V. Then similarly we obtain that U is empty which is a contradiction. Therefore, $\bigcup_{\gamma\in\Gamma} B_r$ is β_0--connected.

Theorem 4.6: If $\{A_n \mid n \in \square\}$ is an infinite sequence of β_0-connected subsets of a topological space X and $A_N \cap A_{n+1} \neq \phi$ for each $n \in \mathbb{N}$, then $\bigcup_{n\in N} A_n$ is β_0-connected.

Proof: The proof follows by induction and Theorem 4.4.

Theorem 4.7: Let X, Y be two topological spaces and $X \times Y$ be with the product topology. If X and Y aere β_0-connected, then $X \times Y$ is β_0-connected.

Proof: For any points (x_1, y_1) and (x_2, y_2) in X×Y, then subspace $(X \times \{y_1\}) \cup (\{x_2\} \times Y)$ contains the two points. This subspace is β_0-connected, since it is the union of two β_0-connected subspaces of X×Y with a point (x_2, y_1) is common. By Theorem 4.4, X×Y is a β_0-connected.

5 β_0-Components

Definition 5.1: Let X be a topological space and x∈X. The β_0-component of X containing x is the union of all β_0-connected subsets of X containing x.

A β_0-component of X is β_0-connected by Theorem 4.4.

Theorem 5.2: For a topological space X the following properties hold:

(1) Each β_0-component of X is a maximal β_0-connected subset of X.

(2) The set of all distinct β_0-components of X forms a portion of X.

(3) Each β_0-component of X is β_0-closed in X.

Proof: (1) Obvious.

(2) Since singleton's are β_0-connected sets, each point x *of* X is contained in the β_0-component of X containing x. Suppose that C_1 and C_2 are two distinct β_0-components of X. If C_1 and C_2 intersects, then $C_1 \cup C_2$ is β_0-connected by Theorem 4.4. Thus, either C_1 is not maximal or C_2 is not maximal, a contradiction. Therefore, C_1 and C_2 are disjoint.

(3) Let C be any β_0-component of X containing x.

By Corollary 4.3, $\beta_0 cl(C)$ is β_0-connected set containing x. Since C is maximal β_0-connected set containing x, $\beta_0$cl(c) ⊆ C. Thus, C is β_0-closed in X.

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